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# A Continuous-Time Model of Financial Clearing

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## Abstract

We present a simple continuous-time model of clearing in financial networks. Financial firms are represented as “tanks” filled with fluid (money), flowing in and out. Once the “pipes” connecting the “tanks” are open, the system reaches the clearing payment vector in finite time. This approach provides a simple recursive solution to a classical static model of financial clearing in bankruptcy, and suggests a practical payment mechanism. With sufficient resources, a system of mutual obligations can be restructured into an equivalent system that has a cascade structure: there is a group of banks that paid off their debts, another group that owes money only to banks in the first group, and so on. We demonstrate that the machinery of Markov chains provides a powerful method to analyze evolution of a deterministic dynamical system.

**Keywords:** Financial networks, clearing vector, continuous time, Markov chain, quasi-linear optimization.

**AMS 2010 subject classification:** Primary 90B10, 90B50, Secondary 60J20.

**JEL Classification:** G21, G33, C61.

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# 1 Introduction

Modern financial systems are richly interconnected networks of various institutions, which depend on each other. With most of the assets of one firm being liabilities of other financial institutions, such a system is naturally prone to “systemic risk” of contagions, in which a default of one firm might lead to defaults of many others (Allen and Gale, 2000; Glasserman and Young, 2016). Optimal regulation of such networks is a subject of intensive debate (e.g., Yellen, 2013).

In a seminal contribution, Eisenberg and Noe (2001) introduced a static framework for the study of interlinked financial systems in distress (see also Suzuki, 2002). They considered an environment in which all participants (hereinafter, “banks”) default within a single clearing mechanism, and demonstrated that there always exists a “clearing payment vector” that satisfies some natural requirements. The Eisenberg-Noe approach has been successfully extended to incorporate liquidity spillovers (Cifuentes, Ferrucci and Shin, 2005; Shin, 2008), outside liabilities (Elsinger, 2009; Glasserman and Young, 2015), costs of default (Rogers and Veraart, 2013), liabilities of different seniority (Kusnetsov and Veraart, 2019), mandatory disclosures (Alvarez and Barlevy, 2015), and other financial instruments, and has become a cornerstone in the analysis of systemic risk (see Hurd, 2016; Feinstein et al., 2018; Kabanov, Mokbel and El Bitar, 2018, for recent surveys).

In this paper, we develop a continuous-time model of financial clearing. We think of interconnected firms as *tanks* filled with money, which flows in and out through pipes connecting the tanks. In the case of default, the pipes open and, given the assumptions about the intensity of the flows consistent with the assumptions of Eisenberg and Noe (2001), the resulting distribution of liquidity in tanks corresponds to the clearing payment vector. Our first set of results is a full characterization of the evolution of the dynamical system, which in turns gives us a finite algorithm to calculate the payment schedule.

Second, evolution of the continuous-time model provides an intuitive explanation for the possible multiplicity of clearing payment vectors. At each moment, we interpret the banking network as a finite Markov chain — despite the fact that our model is deterministic — and study its evolution using the stochastic matrix of current relative liabilities. Groups of banks without any cash that may be involved in mutual reduction of debts,

which we call “swamps”, correspond to ergodic subclasses of the invariant distribution of the Markov chain.<sup>1</sup> These “swamps” are responsible for non-uniquenesses of payment schedules.

Third, our approach provides a simple but intuitive method to reduce many networks of mutual liabilities to an equivalent cascade network in which there is a group of banks that do not have any obligations, the second group of banks that owe money to banks of the first group only, and so on. In these circumstances, there always exists a bank that will repay its obligations in full, even if *all* banks start with a negative amount of cash on hand (outside liabilities). Restructuring that reduces the amount of obligations is practically important, as growing complexity of financial networks creates a significant hurdle for regulators (Yellen, 2013; Bernard, Capponi and Stiglitz, 2017).<sup>2</sup>

Our continuous-time approach to a static problem plays the role of high-speed photography in engineering. It allows us to unpack the process by which the banks that do not have any cash end up repaying all their obligations. At each moment  $t$ , banks are divided into three groups: those that repaid their debts, those that keep paying because they have a positive cash position, and those that keep paying despite having zero cash. Each bank in the latter group pays at the rate equal to the rate at which it receives money from its debtors. In a real crisis, these banks are exactly those that might end up in default, yet would have survived if the problems of sequencing of payments had been resolved as miraculously as in a theoretical model.

The model demonstrates that the clearing mechanism does not require sophisticated planning and hands-on management on behalf of the regulators. Allowing financially constrained firms to repay their debts at the maximum-available speed without any liquidity injections will result in the clearing payment vector. While the existing clearing mechanisms address a number of important practical issues, our results show that, at least theoretically, there is no need in payment-by-payment control in a situation of financial distress. This is especially important given the growing recognition of challenges that

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<sup>1</sup>Though “swamps” might look like a mathematical abstraction, they do appear in the real world. For example, the infamous *Enron* company created special purpose entities that conducted circular transactions, inflating the company’s revenues and hiding the costs (Healy and Palepu, 2003).

<sup>2</sup>Glasserman and Young (2016) cite the FCIC report: “There was no way to know who would be owed how much and when payments would have to be made—information that would be critically important to analyze the possible impact of a Lehman bankruptcy on derivatives counterparties and the financial markets.”

regulators face dealing with extensive and complicated networks of mutual liabilities.

Finally, our results demonstrate the power of the continuous-time approach to static maximization problems. Our continuous-time model generates a discrete dynamical system with the following recursive structure: at each moment, there is a linear system of equations that determines the parameters of the next state of the system and the moment when the system switches to this new state. In a finite number of steps, the system reaches its ultimate state. Thus, the continuous-time model allows us for easy calculation of the payment schedule.

An important advantage of the dynamical model is that it allows us to study different aspects of possible interaction between banks. The model can be extended to deal with financial obligations of multiple maturities in real time, and to be used to take into account stochastic shocks. It might be a convenient tool to study the optimal strategy of a central agency to minimize the potential contagion effect triggered by failure of some banks. In particular, it is straightforward to calculate the minimum amount of cash to be injected into interconnected system to have all banks fully paid off.

[Allen and Gale \(2000\)](#) and [Freixas, Parigi and Rochet \(2000\)](#) pioneered studies of the role of interconnectedness in the stability of financial networks. [Rogers and Veraart \(2013\)](#) extended [Eisenberg and Noe \(2001\)](#) by introducing the possibility of a partial default, and described an algorithm that results in the largest clearing vector for any environment (see also [Amini, Filipović and Minca, 2016](#)). [Battiston et al. \(2012\)](#) demonstrated that a higher diversification of each individual network member might lead to an increase, rather than a decrease, of the systemic risk. [Liu and Staum \(2010\)](#) provided a formula for sensitivity analysis of Eisenberg and Noe’s one-period model of contagion via direct bilateral links. [Feinstein et al. \(2018\)](#) quantified the Eisenberg–Noe clearing vector’s sensitivity.

In an empirical study, [Azizpour, Giesecke and Schwenkler \(2018\)](#) demonstrated that contagions, through which the defaulting firms have a direct impact on the financial health of their counterparts, is a significant source of default clustering. [Banerjee, Bernstein and Feinstein \(2020\)](#) generalized the model to allow for continuous-time dynamics of the interbank liabilities using stochastic differential equations, proved the existence of a solution for this generalized model, and analyzed the properties of this solution. [Csóka and Herings \(2018\)](#) used the Eisenberg–Noe approach to analyze decentralized clearing.

Capponi, Corell and Stiglitz (2020) applied it to study the optimal policy in the case of a sovereign default.

There is substantial literature that focuses on the specific forms of networks connecting financial firms. Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) demonstrated the non-monotonic effect of the magnitude of negative shocks on network stability. The network density contributes to stability when the shocks are relatively small, but propagate contagion at a certain magnitude. Castiglionesi and Eboli (2018) show that the star-shaped network is most resilient to systemic risks. In this paper, we take the network as given, and our results apply to all networks. Our model highlights that a restructuring of a system of mutual obligations might lead to another, simpler system, which is equivalent in terms of the ultimate clearing vector.

Eboli (2013, 2019) analyze the impact of shock to a financial system using the notion of the flow network.<sup>3</sup> The analysis in Eboli (2013, 2019) is confined to complete, star, and ring networks: for these networks, there exists a unique function that associates the financial losses transmitted by defaulting debtors to their creditors to the links of a network. Our approach works for any finite network; in fact, we do not have any references to the network structure in our results.

Veraart (2020) uses a network model to describe how contagions, including *distress contagions*, which precede, rather than follow, bankruptcy, spread through the system. Elliott, Golub and Jackson (2014) analyze the role of integration and diversification in the prevention of cascade failures. Amini, Cont and Minca (2016) study the resilience of a large financial network to defaults using contagion in random graphs. Feinstein (2017) generalized Amini, Filipović and Minca (2016) by allowing for differing liquidation strategies, and provided sufficient conditions for the existence of an equilibrium liquidation strategy with a corresponding unique clearing vector. Cifuentes, Ferrucci and Shin (2005) note that the full connectedness of the financial network is a sufficient condition for the uniqueness of the clearing vector (see also Shin, 2010). Our approach allows to refine this observation: it is the presence of ergodic subclasses among banks that have zero cash at the initial moment that generates multiplicity. (Eboli 2019, and Roukny, Battiston and Stiglitz 2018, identify these sources of multiplicity using very different approaches.)

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<sup>3</sup>Though Eboli (2013) is a working paper version of Eboli (2019), the two papers use somewhat different refinement technique and are of independent interest.

The rest of the paper is organized as follows. In Section 2, we use the original Eisenberg and Noe (2001) model to discuss the main questions that our continuous-time model addresses. Section 3 introduces the continuous-time setup. In Section 4, we prove the existence theorem and characterize the dynamic process that leads to the clearing vector. In Section 5, we use Markov chains to characterize the sources of multiplicity of clearing vectors. In Section 6, we focus on models that can be restructured into the cascade form. Section 7 discusses extensions. Section 8 concludes. Appendix contains technical details.

## 2 Eisenberg and Noe Framework

In this section, we outline the original Eisenberg and Noe (2001) model, use it to discuss the main questions that our continuous-time model seeks to answer, and describe informally the answers our model provides.

**Eisenberg and Noe (2001).** There is a financial system composed of  $n$  banks (firms, financial institutions, agents, etc.) owing money to each other, which is described by the matrix of mutual liabilities  $B = \{b_{ij}\}$ . Vector  $\mathbf{c} = (c_1, c_2, \dots, c_n)$  describes the initial cash positions  $c_i \geq 0$  for all  $i$ . The matrix of mutual liabilities  $B$  uniquely determines the *debt vector*  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  and the *stochastic matrix of relative liabilities*  $Q = \{q_{ij}\}$ , so that  $b_{ij} = b_i q_{ij}$  for every  $i, j$ . Vice versa, matrix  $Q$  and vector  $\mathbf{b}$  uniquely determine the matrix of liabilities  $B$ . In our continuous-time model, they will be functions of time.

The *clearing vector*  $\mathbf{p} = (p_1, p_2, \dots, p_n)$ , where  $p_i$  is the total amount paid by bank  $i$  to all other banks, should satisfy the following two conditions (Eisenberg and Noe, 2001):

(A) *Priority and proportionality of debt claims.* With total payment  $p_i$ , the bank  $i$  pays to  $j$  the fraction  $p_i q_{ij}$  in such a way that either its total debts are paid or all of its resources are exhausted.

(B) *Limited liability.* The total payment of any bank should never exceed the cash flow available to the bank, i.e., the initial cash plus the money received from other banks.

Conditions (A) and (B), taken together, imply that the clearing vector  $\mathbf{p}$  satisfies

$$\mathbf{p} = \min(\mathbf{c} + Q^T \mathbf{p}, \mathbf{b}). \tag{1}$$

In all equations,  $\mathbf{p}$ ,  $\mathbf{c}$ , and  $\mathbf{b}$  are column  $n$ -dimensional vectors, the minimum is taken component-wise,  $T$  is the transposition symbol, and thus  $(Q^T \mathbf{p})_i$  equals the total amount received by bank  $i$  from other banks. The existence of such a vector  $\mathbf{p}$  follows, e.g., from the Knaster–Tarski lattice version of the fixed-point theorem, but in our model its existence will follow almost immediately from the structure of our model. If  $p_i < b_i$ , bank  $i$  cannot repay its obligations.

Our Theorem 1 shows that if the system starts with the debt and cash vectors as in Eisenberg and Noe (2001), then, with “pipes” open and money flowing between “tanks”, the outcome is exactly the clearing vector. It also gives a full characterization of a discrete dynamical system that can be solved via a straightforward algorithm.

If all banks start the procedure with some cash on hand, the solution path is unique. Otherwise, there might be complications. Note that condition (A) is a strong requirement: it implies that if, e.g.,  $b_{ij} = 1$  and  $b_{ji} = 2$ , then these banks cannot “cancel” these liabilities by replacing them with  $b_{ij} = 0$  and  $b_{ji} = 1$ , since the proportionality of debt claims will be violated. Another observation is that even if all  $c_i = 0$ , there may exist nontrivial solutions of equation (1), and we will show that this is exactly the cause of potential multiplicity of solutions.

**The Transactions Problem.** The main focus of the original Eisenberg and Noe (2001) was on the existence of the clearing vector  $\mathbf{p}$ , a solution to the fundamental clearing equation (1). Yet even when the clearing vector  $\mathbf{p}$  is known, how will the banks pay their liabilities? How could they do this simultaneously? How should the banks that are ultimately solvent pay their debts *before* they receive money from their counterparts? What is the optimal sequence of payments between banks that results in the clearing vector  $\mathbf{p}$ ?

To address these questions, let us introduce a few definitions. Given a model  $M = (\mathbf{b}, Q, \mathbf{c})$ , we will have, in addition to the clearing vector  $\mathbf{p}^*$ , i.e., *final payments*, the vector of *amounts received*  $\mathbf{r}^* = Q^T \mathbf{p}^*$ , the *final cash positions*  $\mathbf{c}^* = \mathbf{c} + \mathbf{r}^* - \mathbf{p}^*$ , the *unpaid debts*  $\mathbf{b}^* = \mathbf{b} - \mathbf{p}^*$ , and the *losses*  $\mathbf{l}^* = Q^T (\mathbf{b} - \mathbf{p}^*)$ .

We also have the following balance conditions:

$$\begin{aligned}
\sum_i c_i &= \sum_i c_i^*, \\
\sum_i b_i &= \sum_i (Q^T b)_i, \\
\sum_i p_i^* &= \sum_i (Q^T p^*)_i, \\
\sum_i l_i^* &= \sum_i (b_i - p_i^*).
\end{aligned} \tag{2}$$

The first three equations hold since the model is self-contained. The last equality follows from the previous equations.

In contrast with [Eisenberg and Noe \(2001\)](#), we allow the vector of initial cash positions  $\mathbf{c} = (c_1, \dots, c_n)$  to have negative entries. Thus, it makes sense to introduce the *vector of minimum cash positions*  $\mathbf{c}^m(\mathbf{b}, Q)$ , the minimum vector such that all banks should be able to pay their debts. If this happens,  $\mathbf{p}^* = \mathbf{b}$ ,  $\mathbf{c}^* = (0, \dots, 0)$ , and the two terms on the right side of the clearing equation (1) must coincide, i.e.,  $\mathbf{b} = \mathbf{c} + Q^T \mathbf{p}^*$ .

Using the clearing equation, we immediately obtain simple but useful results about the vector of minimum cash positions  $\mathbf{c}^m$ .

First, this vector satisfies

$$\mathbf{c}^m = (I - Q^T)\mathbf{b}, \tag{3}$$

Second,

$$\sum_{i=1}^n c_i^m = 0. \tag{4}$$

Third, if the initial cash vector  $\mathbf{c}$  satisfies  $\mathbf{c} \succ \mathbf{c}^m$ , i.e., the former vector is *larger* (component-wise) than the latter, then  $\mathbf{c}^* = \mathbf{c} - \mathbf{c}^m$  and the clearing vector satisfies  $\mathbf{p}^* = \mathbf{b}$ . Otherwise, at least one bank will be unable to repay its debts.

In the case when the clearing vector  $\mathbf{p}^* = \mathbf{b}$ , we say that the model is *sufficient*, otherwise, it is *deficient*. Obviously, vector  $\mathbf{c}^m$  is the minimum vector that makes the model sufficient. Thus, with the same parameters  $Q$  and  $\mathbf{b}$ , the model will be sufficient or deficient depending on the cash vector  $\mathbf{c}$ . The difference between the two cases is

substantial: in the former case, the pairwise mutual obligations can be cancelled; in the latter they cannot. To answer whether vector  $\mathbf{c}$  is sufficient or deficient, there is no need to solve the clearing equation: we only need to compare this vector with vector  $\mathbf{c}^m$ , which can be calculated using (3).

**Example 1.** Let  $n = 3$  and matrix  $Q$  be given by  $q_{12} = q_{13} = \frac{1}{2}$ ,  $q_{21} = \frac{1}{3}$ ,  $q_{23} = \frac{2}{3}$ ,  $q_{31} = \frac{1}{4}$ , and  $q_{32} = \frac{3}{4}$ :

$$Q = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/4 & 3/4 & 0 \end{pmatrix} \quad (5)$$

With this matrix, consider the debt vector  $\mathbf{b} = (12, 21, 20)$ . For  $\mathbf{b}$ , using formula (3), we obtain  $\mathbf{c}^m(Q, \mathbf{b}) = (0, 0, 0)$ . That is, to pay all the debts, it suffices that no bank has negative cash at the initial moment.

If the cash vector is sufficient, i.e.,  $\mathbf{c} \succ \mathbf{c}^m$ , then the clearing (payment) vector  $\mathbf{p}^*$  coincides with the debt vector  $\mathbf{b}$ . In the case of  $\mathbf{b} = (12, 21, 20)$  in Example 1, *no transactions are necessary*. The job of a clearing house in this case is just to inform the banks that their obligations cancel each other. The natural question is as follows: given matrix  $Q$ , for which debt vector  $\mathbf{b}$  is the minimum cash vector  $\mathbf{c}^m = (0, 0, 0)$ ? As we will see, the necessary and sufficient condition is that the debt vector  $\mathbf{b}$  is proportional to the invariant distribution  $\pi$  of the corresponding ergodic Markov chain with matrix  $Q$ . In Example 1,  $\pi = \frac{1}{53}(12, 21, 20)$ . The general result is given in Theorem 2.

**Cascade Structure of Liabilities.** The minimum initial cash vector  $\mathbf{c}^m$  that satisfies (3) answers the following question. What is the smallest initial cash vector  $\mathbf{c}$  such that *all banks* ultimately pay their debts? Now, given matrix  $Q$  and vector  $b$ , what is the smallest vector  $\mathbf{c}$  such that *at least one bank* will pay its debt? The answer to this question might seem surprising: a bank that fully repays its debt exists for almost any initial cash vector  $\mathbf{c} = (c_1, \dots, c_n)$ , even if most of  $c_i < 0$ . Indeed, (4) implies that if  $c_i^m > 0$  for at least one bank, then for at least one other bank  $c_j^m < 0$ , i.e., this bank  $j$  will ultimately be a net recipient and can start with a negative cash position. If  $\mathbf{c}^m = (0, \dots, 0)$ , then the mutual obligations of all banks cancel each other, and thus no real transactions are necessary.

Let us go through the actual payment mechanism in the next example.

**Example 2.** Consider the same matrix  $Q$  given by (5), but a different debt vector,  $\mathbf{b} = (13, 22, 20)$  (see also Figure 1(a)). Using formula (3), we obtain the minimum cash vector  $\mathbf{c}^m(Q, \mathbf{b}) = (\frac{2}{3}, \frac{1}{2}, -\frac{7}{6})$ : as we will see, even with a negative initial cash position, bank 3 will be able to pay its debts in full. First, let us demonstrate that the debt vector  $\mathbf{b} = (13, 22, 20)$  is equivalent to the debt vector  $(1, 1, 0)$ . Suppose bank 1 prepares payments of 6 to banks 2 and 3, bank 2 prepares a payment of 7 to bank 1 and a payment of 14 to bank 3, and bank 3 prepares a payment of 5 to bank 1 and a payment of 15 to bank 2. These *potential* payments satisfy the proportionality assumption (A) but formally violate assumption (B). Yet these potential payments partially cancel each other: e.g., 6 from bank 1 to bank 2, and 7 from bank 2 to bank 1 result in just 1 owed by bank 2 to bank 1. These potential cancellations plus a cancellation of a circular debt of 1 result in the remaining debt vector  $(1, 1, 0)$ . Now, given matrix  $Q$ , bank 3 has no debt, bank 1 owes  $\frac{1}{2}$  to bank 3 and  $\frac{1}{2}$  to bank 2, and bank 2 owes  $\frac{1}{3}$  to bank 1 and  $\frac{2}{3}$  to bank 3. If the initial cash vector is the minimum vector  $\mathbf{c}^m = (\frac{2}{3}, \frac{1}{2}, -\frac{7}{6})$ , then at step 1, bank 1, having  $\frac{2}{3}$  as its initial holding, sends  $\frac{1}{2}$  to bank 3 and  $\frac{1}{6}$  to bank 2; at step 2, bank 2, having  $\frac{1}{2}$  from its initial holding and  $\frac{1}{6}$  obtained from bank 1 at step 1, sends  $\frac{1}{3}$  to bank 3. All debts are paid, and the final cash position is  $\mathbf{c}^* = (0, 0, 0)$ .

We call a structure of obligations and corresponding payments a *cascade* structure, or simply the model a cascade, if set  $J$  can be partitioned into sets  $(J_1, J_2, \dots, J_m)$  such that the structure of obligations has the following form: there is a group of banks  $J_1$  without any debts, there is a group of banks  $J_2$  that owe money only to banks in  $J_1$ , etc., and, finally, there is a group of banks  $J_m$ ,  $m \leq n$ , to which no bank owes money, and these banks may have debts to all other banks. To simplify presentation in all examples, we consider the situation when each set  $J_k$  has only one bank, and then  $m = n$ . Figures 1(b) and 1(c) show the order of transfers in two cascades described in Example 2, where the cascade 1(b)  $(J_1, J_2, J_3)$  is  $(3, 2, 1)$ . Note that the payment steps go in inverse order: banks 1 pays to banks 2 and 3, then bank 2 pays to 3. The reason for the order of sets in the cascade follows from the algorithm of cascade construction that first finds set  $J_1$ , then  $J_2$ , and so on.

Surprisingly, even in this very simple example, the transaction sequence is not unique

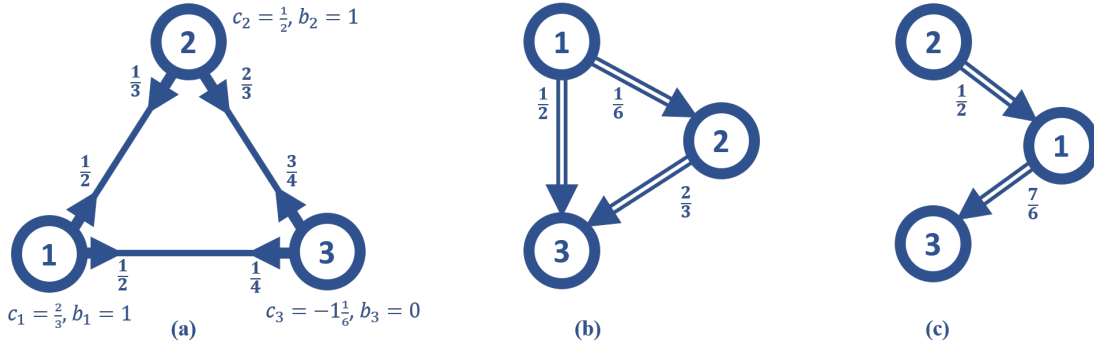


Figure 1: (a) The matrix of relative liabilities and the initial cash vector from Example 2. (b) Transfers in cascade 1 result in full repayment. (c) Transfers in alternative cascade 2 result in full repayment as well.

— not only in the timing of the payments, but also in the order of the states in the cascade. Let us obtain a second cascade using another mechanism of debt restructuring for the same model and the sufficient cash vector (see Figure 1(c)). For a sufficient model, the pairwise obligations can be reduced to unilateral ones: bank 1 owes  $\frac{3}{2} = 13 \times \frac{1}{2} - 20 \times \frac{1}{4}$  to bank 3, bank 2 owes  $\frac{5}{6} = 22 \times \frac{1}{3} - 13 \times \frac{1}{2}$  to bank 1, and bank 3 owes  $\frac{1}{3} = 20 \times \frac{3}{4} - 22 \times \frac{2}{3}$  to bank 2. Since these obligations form a closed circuit, the amount  $\frac{1}{3}$  can be subtracted and now the obligations are: bank 3, as in the first cascade, has no debts, bank 1 owes  $\frac{7}{6}$  to bank 3, and bank 2 owes  $\frac{1}{2}$  to bank 1. Then the second cascade 1(c) is (3, 1, 2), i.e., at step 1, bank 2, having  $\frac{1}{2}$ , sends  $\frac{1}{2}$  to 1; at step 2, bank 1, having  $\frac{2}{3}$ , adds  $\frac{1}{2}$  obtained at step 1 and sends  $\frac{7}{6}$  to bank 3. Again, all debts are paid, and the final cash position is  $\mathbf{c}^* = (0, 0, 0)$ .

Theorem 3 in Section 6 establishes that any sufficient system could be modified, via a string of admissible payments, to an equivalent system that has a cascade structure. In Subsection 6.2, we describe a recursive algorithm to construct the cascade of type 1(b).

### 3 Setup

In our model, the information is summarized by the triplet  $M = (\mathbf{b}, Q, \mathbf{c})$ , where  $\mathbf{b} = (b_1, \dots, b_n)$  is the debt vector, the stochastic matrix  $Q = \{q_{ij}\}$  describes the relative liabilities, and the vector of initial cash positions  $\mathbf{c} = (c_1, \dots, c_n)$  has the same meaning as in the original Eisenberg and Noe model, with one important distinction: we allow  $c_i \leq 0$

for some or even all  $i$ .<sup>4</sup> The matrix of liabilities  $B = \{b_{ij}\}$  uniquely determines matrix  $Q = \{q_{ij}\}$  and the debt vector  $\mathbf{b}$ , and vice versa:  $b_{ij} = b_i q_{ij}$  for any  $i, j$ . By convention, if for some bank  $b_i = 0$ , yet there is  $k$  with  $q_{ki} > 0$ , then we define  $q_{ii} = 1$  and  $q_{ij} = 0$  for all  $j \neq i$ .

Since the initial cash positions might be negative, we will appropriately modify equation (1): we require  $\mathbf{p}$  to satisfy

$$\mathbf{p} = \max\{\min(\mathbf{c} + Q^T \mathbf{p}, \mathbf{b}), 0\}. \quad (6)$$

We assume that the conditions (A) and (B) are always satisfied.

In our model, the parameters that describe each bank  $i$ , i.e.,  $b_i, c_i$ , and  $p_i$ , depend on time that runs on a finite interval starting with  $t = 0$ . The initial position of bank  $i$  is  $x_i(0) = (b_i, c_i)$ . The position of bank  $i$  at time  $t$  is described by the vector  $x_i(t) = (b_i(t), c_i(t))$ , representing its *remaining* debt and its *current* cash position, and the system as a whole is described by a  $2n$ -dimensional vector  $\mathbf{x}(t) = (x_i(t), i = 1, \dots, n)$ . The amount that bank  $i$  has paid by moment  $t$ ,  $p_i(t) = b_i - b_i(t)$ , will be later represented as an integral of a function  $u_i(s)$ , the rate at which bank  $i$  pays its debt at moment  $s$ . This rate, which we call the *out-rate*, will be the main control parameter in our model.

One can visualize the continuous-time model as follows. Each bank  $i$  is a tank filled with a fluid (money) with initial level  $c_i = c_i(0)$ , which might be positive, zero, or negative. Each tank is connected to all other tanks by *inlet* and *outlet* pipes. The *maximum possible rate* of the flow through each pipe  $(i, j)$ , which is an outlet for  $i$  and an inlet for  $j$ , is its *capacity*  $q_{ij}$ . Since matrix  $Q$  is stochastic, the total capacity of all outlets from tank  $i$  is 1. Bank  $i$  pays its debts at the out-rate of  $u_i(t)$ , the intensity with which money flows from tank  $i$  to other tanks. Thus, at any point in time, bank  $i$  pays to bank  $j$  at the rate of  $q_{ij}u_i(t)$ , proportional to its total debt under assumption (A).

To fully specify the continuous-time model, we add assumptions (C1) and (C2) to assumptions (A) and (B).

**(C1)** *At any moment  $t$ , all out-rates must satisfy  $0 \leq u_i(t) \leq 1$ .*

The out-rates  $u_i(t)$  completely specify the in-rates  $n_i(t) = \sum_{j \in J} u_j(t)q_{ji}$ , which allows

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<sup>4</sup>Sometimes vector  $\mathbf{c}$  is interpreted as a vector of outside assets; a negative entry corresponds to outside liabilities (Elsinger, 2009; Glasserman and Young, 2016).

us to focus on the former. Note that since the in-rates  $n_i$  are defined by the *columns* of the stochastic matrix  $Q$ , they can potentially be less or greater than one.

To formulate assumption (C2) about in-rates  $u_i(t)$  for different banks, we need to introduce two other interrelated notions, which will play a key role in our analysis. Define a *partition* of the set of all banks  $J$  into three groups,  $P(t) = (J_+(t), J_0(t), J_*(t))$ . A partition is a function of time  $t$ ; that is, a bank belongs to one of the three groups at each moment  $t \geq 0$ . The first group of banks  $J_+(t)$  is called *positive* and consists of those banks that, at time  $t$ , still have outstanding debt and a positive cash position. The out-rates in this group can have any values  $u_i(t)$ ,  $0 \leq u_i(t) \leq 1$ . The next group  $J_0(t)$ , called *zero*, are those banks that have positive debt and a zero or even a negative cash position. They might be still paying their debts with money flowing to them from other banks. The out-rates in this group must coincide with the in-rates,  $u_i(t) = n_i(t)$ , at any moment in time. The last group,  $J_*(t)$ , called *paid-up*, includes those banks that have paid out all of their debts and are still possibly receiving money from other banks. For them, the out-rates are equal to zero,  $u_i(t) = 0$ .

Summing up, at each moment  $t$ , we have a partition of all banks into three groups:

$$\begin{aligned} J_+(t) &= \{i : b_i(t) > 0, c_i(t) > 0\}; \\ J_0(t) &= \{i : b_i(t) > 0, c_i(t) \leq 0\}; \\ J_*(t) &= \{i : b_i(t) = 0\}. \end{aligned} \tag{7}$$

Being in one of these groups determines the bank's *status* at moment  $t$ .

Assumption (C2) specifies out-rates  $u_i(t)$  for all banks at all times  $t \geq 0$ :

$$\begin{aligned} \text{(C2)} \quad & \text{If } i \in J_+(t), \text{ then } u_i(t) = a_i, \quad 0 < a_i \leq 1; \\ & \text{If } i \in J_*(t), \text{ then } u_i(t) = 0. \\ & \text{If } i \in J_0(t), \text{ then } u_i(t) = n_i(t) = \sum_{j \in J_+(t)} a_j q_{ji} + \sum_{j \in J_0(t)} u_j(t) q_{ji}. \end{aligned} \tag{8}$$

We prove the existence of functions  $u_i(t)$  in Theorem 1, and give necessary and sufficient conditions for uniqueness of the solution in Theorem 2. We denote  $d_i(t) = n_i(t) - u_i(t)$ , the *balance rate*, since this coefficient defines whether cash position  $c_i(t)$

will grow, decline, or remain constant. For the zero group, the out-rates must be equal to the in-rates, so  $d_i(t) = 0$ . We call banks with a positive or a zero status, i.e., banks continuing to pay their debts, *active*, and other banks *nonactive*. We say that bank  $j$  is a *sponsor* of bank  $i$  at moment  $t$  if it pays its debt to  $i$ , i.e., if  $u_j(t)q_{ji} > 0$ .

For any subset  $B \subseteq J$  of the set of all banks  $J = \{1, 2, \dots, n\}$ ,  $Q_B$  will denote the matrix obtained from the stochastic matrix  $Q$  by deleting all rows and columns not in  $B$ ;  $I_B [= I]$  denotes the identity matrix of a corresponding dimension. Vector  $v_B$ , derived from the  $n$ -dimensional vector  $v$ , is defined similarly.

The choice of positive constants  $a_i$  is a matter of convenience. For the selection of  $a_i$  for the positive group of banks we will use only two options. If set  $J_0(0)$  is empty, we assume that all  $a_i = 1$ . We call this a *regular* case and refer to this flow as the *basic flow*. For example, if  $c_i = c_i(0) > 0$  for every bank  $i$ , we are dealing with a regular case.

The only other values of constants  $a_i$  that we are going to use are  $a_i = \pi_i > 0$  for  $i \in B \subseteq J$ , where  $\pi_i$  are given by the invariant distribution of the corresponding Markov chain if matrix  $Q_B$  is ergodic. We call them the *invariant rates*. These are the only rates for positive banks when their in-rates coincide with the out-rates, and therefore their cash positions are constant.

If there exists a bank  $i$  such that  $c_i = c_i(0) \leq 0$ , the solution to (6) is not necessarily unique. We deal with this case in Section 5. Theorem 2 characterizes the parameters for which there is a multiplicity of solutions. In addition, there is also a possibility that for some banks with  $c_i(0) = 0$ , the in-rates at the *initial* moment may exceed 1, and thus some of these banks should be “instantly” reclassified as positive, since they will have  $c_i(t) > 0$  on some nonzero interval  $(0, \delta)$ . We will discuss the so-called Big Bang effect in Subsection A2.<sup>5</sup>

## 4 The Regular Case

In this section, we start with stating formally our Theorem 1 that focuses on the case when all banks have a positive amount of cash at the initial moment:  $c_i > 0$  for all  $i$ , i.e., the set of zero banks  $J_0(0) = \emptyset$  and the basic out-rates  $u_i = 1$  are used for the

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<sup>5</sup>For a similar reason, the Big Bang cosmology assumes that the life of the Universe starts after the “Planck epoch,” a minimum period of time, has passed.

positive group. We then describe how the evolution of the continuous-time model results in an Eisenberg-Noe clearing vector in finite time and characterize the discrete dynamical system that describes the evolution, completing the proof.

## 4.1 The Existence Theorem

What happens when all pipes are open at time 0? The assumptions (A), (B), (C1), and (C2) jointly determine the following dynamics:

$$\begin{aligned} p_i(t) &= \int_0^t u_i(s) ds, \\ b_i(t) &= b_i - p_i(t), \\ c_i(t) &= c_i + \sum_j p_j(t) q_{ji} - p_i(t) = c_i + \int_0^t d_i(s) ds, \end{aligned} \tag{9}$$

where  $d_i(s) = n_i(s) - u_i(s)$  is the balance rate and  $n_i(s) = \sum_j u_j(s) q_{ji}$  is the in-rate for bank  $i$  at moment  $s$ .

The money will flow while there is at least one bank with a positive status, i.e., having an outstanding debt and a positive cash reserve. Before that moment, the total amount of debt in the system is monotonically decreasing at least at the unit rate, and thus, since the total amount of debt is finite, the process ends at a finite time  $T_* = \min \{t : J_+(t) = \emptyset\}$ . Theorem 1 describes the main characteristics of this process for a regular initial vector  $\mathbf{c}$ .

**Theorem 1.** *(a) For any regular initial vector  $\mathbf{c}$ , there exists a finite time  $T_* = \min \{t : J_+(t, \mathbf{c}) = \emptyset\}$  at which all out-rates  $u_i(T_*) = 0$  for every bank  $i$ . The time interval  $[0, T_*)$  consists of a finite number  $k_*$  of half-intervals  $\Delta_k = [T_k, T_{k+1})$ ,  $k = 1, 2, \dots, k_*$ ,  $T_1 = 0$ ,  $T_{k+1} = T_*$ . The out-rates  $u_i(t)$  are constant on these intervals; functions  $p_i(t)$ ,  $b_i(t)$ ,  $c_i(t)$ ,  $0 \leq t \leq T_*$ , satisfying equations (9), are correspondingly linear on each interval  $\Delta_k$ . The moments  $T_k > 0$  are the moments when at least one bank changes its status, i.e., one of the functions  $b_i(t)$ ,  $c_i(t)$  hits zero from above. Vector  $\mathbf{p}(T_*)$  solves equation (6), i.e., it is the clearing payment vector in both the Eisenberg-Noe and continuous-time models.*

*(b) The following Monotonicity Property holds: in-rates  $n_i(t)$  are nonincreasing in  $k$  for all  $i \in J$ , i.e.,  $n_i(T_k) \geq n_i(T_{k+1})$ ,  $k = 1, 2, \dots, k_* - 1$ . This implies a similar property*

for out-rates  $u_i(t)$ .

(c) Partition  $P_k = P(T_k) = (J_{+,k}, J_{0,k}, J_{*,k})$ , and vectors  $\mathbf{b}_k = (b_i(T_k))$  and  $\mathbf{c}_k = (c_i(T_k))$  contain all the information about the system at moment  $T_k$  and uniquely define the out-rates  $u_{i,k}$  (and thus all other functions) on the next interval  $\Delta_k$ .

Because the evolution of the system is interesting in itself, we will prove parts (a) and (c) of Theorem 1 in the next subsection, and delegate a more technical proof of part (b) to the Appendix.

## 4.2 Evolution in Continuous Time

When the process ends at  $T_* = \min \{t : J_+(t) = \emptyset\}$ , we have the following picture. As the first of the balance conditions (2) shows, the sum of all cash positions is constant at all times. Thus, at the final moment  $T_*$ , we have

$$\sum_i c_i = \sum_i c_i(T_*) = \sum_{i \in J_*(T_*)} c_i(T_*).$$

The last equality holds because  $J_+(T_*) = \emptyset$  and  $c_i(T_*) = 0$  for all  $i \in J_0(T_*)$ . These equalities immediately imply that the set  $J_*(T_*)$  is always non-empty even if all  $c_i \ll b_i$ , and the sum  $\sum_i c_i$  is positive.

What happens before this final moment? According to Assumption (C2), all out-rates, and hence the in-rates as well, are constant on any interval where the statuses are not changed, and therefore all cash positions, debts, and payments are linear functions on these intervals. Thus, in the regular case, when all  $c_i(0) > 0$ , all banks have positive status until some moment  $T_1$  when either one debt or cash level hits zero. This holds because all debts decrease at a unit rate. Without loss of generality, perturbing slightly, if necessary, the initial conditions, we can assume that at the moments of status change only one bank changes its status.

At any moment  $t$ , a status change can occur only when some positive bank pays up its debt, or its cash position hits zero, or some zero bank pays up its debt. In the first case, the zero group remains the same, and the in-rates for all banks, including the solution of (8) for the zero group, do not increase. The equality  $c_i(t) = c_i$  holds for each bank in the zero group. In the second case, when one positive bank changes its status to zero,

this is possible only when the balance rate  $d_i$  of this bank is negative, i.e., the in-rate  $n_i$  was less than the out-rate  $u_i = 1$ . This bank is added then to the zero group, and the out-rate of this bank drops from  $u_i = 1$  to the new value  $n_i < 1$ , defined by the new solution of (8). The in-rates for other banks in the zero group also decrease, since the total contribution to this group from the positive sponsors decreases. In the third case, the situation is similar.

Thus, the in-rates are always non-increasing for all banks. The formal proof that the in-rates in the zero group, and in all groups, no matter if the zero group is expanding or contracting, are non-increasing, i.e., point (b) of Theorem 1, is given in the Appendix.

Assuming that point (b) of Theorem 1 holds, the interval  $[0, T_*)$  consists of a finite number  $k_*$  of half-intervals  $\Delta_k = [T_k, T_{k+1})$ ,  $k = 1, 2, \dots, k_*$ ,  $T_0 = 0$ , where the times  $T_k > 0$  are the only times when one bank changes its status, i.e., one of the functions  $b_i(t)$ ,  $c_i(t)$  hits zero, having been positive before. Plus, (b) yields an important *irreversibility property* for a status change after the initial moment: *The only possible status change for all  $t > 0$  is: from the positive group to the paid-up group, or to the zero group, and from the zero group to the paid-up group.*

Because of the irreversibility, the number of moments  $T_k$  is no greater than  $2n$ . For the sake of brevity, we call the intervals  $\Delta_k$  *status intervals*. Note that in the general situation when at moment  $t = 0$  there are banks with  $c_i \leq 0$ , an instant status change is possible. This is due only to possible reclassification of some zero banks to positive (see the Bang Bang effect in Subsection A2).

To complete the proof of (a) in Theorem 1, we will need the following slightly less straightforward result.

**Lemma 4.1.** *The amounts paid by each bank at moment  $T_*$ , i.e.,  $p_i = p_i(T_*)$ ,  $i \in J$ , give a solution to the clearing equation (6).*

*Proof.* The status of each bank  $i$  at moment  $T_*$  is either zero or paid-up. If it is paid-up, then the total payment of this bank coincides with its debt, i.e.,  $p_i(T_*) = b_i$ . If bank  $i$  at  $T_*$  has a zero status, i.e.,  $c_i(T_*) = 0$ , the debt  $b_i$  is not fully paid,  $p_i(T_*) < b_i$ . Since the rate of payment of each bank  $j$  to bank  $i$  at each time  $t$  is proportional to capacity  $q_{ji}$ , the total payment obtained by bank  $i$  from bank  $j$  is  $p_j(T_*)q_{ji}$ . Using the equality  $c_i(T_*) = 0$  and the second of the balance equalities (2), we obtain that the total payment

of bank  $i$  is equal to the sum of the initial cash  $c_i$  and the money obtained from other banks, i.e.,  $p_i(T_*) = c_i + \sum_j p_j(T_*)q_{ji} < b_i$ . Thus, in both cases the amount paid by each bank satisfies the clearing equation.  $\square$

### 4.3 The Discrete Dynamical System

Our next goal is to describe all information needed to compute the end time of each status interval  $\Delta_k$  and complete the proof of part (c) of Theorem 1.

According to our definitions and part (a) of Theorem 1, we have continuous flow  $X(t)$  defined on the interval  $[0, T_*)$  by vector  $X(t) = (t, P(t), b_i(t), c_i(t), i \in J)$ . On each status interval, all out-rates are constant and define a simple linear dynamics for all parameters of the model. As a result, we can utilize one of the main computational advantages of our model: its dynamics is fully characterized by a homogeneous *discrete dynamical system*  $(X_k)$ , where time moments  $k = 1, \dots, k_* + 1$  correspond to moments  $T_k$  in the continuous-time model, and vector  $X_k$  contains all information about the system at this moment. This vector is  $X_k = (T_k, P_k, b_{i,k} = b_i(T_k), c_{i,k} = c_i(T_k), i \in J)$ , where  $P_k = P(T_k)$  is a partition of the system at the beginning of the next interval  $\Delta_k = [T_k, T_{k+1})$ .

When it is clear which interval  $\Delta_k$  is considered, to simplify notation, we suppress further indices  $k$ , denoting  $\Delta_k = \Delta$ ,  $T_k = T$ ,  $T_{k+1} = T'$ , the out-rates  $u_{i,k} = u_i$ , etc. The values of the out-rates for the basic flow are selected at moment  $T_k$  for the next interval  $\Delta_k$  according to (C2), and partition  $P_k = P(T_k) = (J_+, J_0, J_*)$ , i.e.:  $u_i = 1$  if  $i \in J_+$ ,  $u_i = 0$  if  $i \in J_*$ , and if  $i \in J_0$ , then  $u_i = v_i$ , where vector  $v = (v_i)$  is obtained as a solution of a linear non-homogeneous system of equations (8). Thus, we have

$$\begin{aligned} b_i(t) &= b_i(T_k) - u_i t, \\ c_i(t) &= c_i(T_k) + (n_i - u_i)t. \end{aligned} \tag{10}$$

The time of the next status change  $T' = T_{k+1}$  is the first time when one of those quantities  $b_i(t), c_i(t)$ , which were positive at moment  $T = T_k$ , will reach zero for the first time. Assuming that the out-rates on interval  $(T, T + t)$  are equal to  $u_i$ , we denote the *potential* moment  $T + s_i$  when bank  $i$  will repay its debt, and the *potential* moment  $T + t_i$  when

bank  $i$  reaches the zero cash position. Equations (10) immediately imply that

$$s_i = \begin{cases} b_i(T)/u_i, & \text{if } b_i(T) > 0, \text{ and } u_i > 0; \\ +\infty & \text{if } u_i = 0. \end{cases}$$

$$t_i = \begin{cases} -c_i(T)/d_i, & \text{if } c_i > 0, \text{ and } d_i < 0; \\ +\infty & \text{if } c_i \leq 0 \text{ or } d_i \geq 0. \end{cases}$$

Finally,

$$T' = T + \min_i(s_i, t_i). \quad (11)$$

Note that for  $i \in J_0$ , we have  $c_i(t) \equiv c_i(T) \leq 0$  for all  $t \in (T, T')$ , so for them  $t_i = +\infty$ .

Thus, state  $X_k$  with the out-rates specified by (C2), (8), and a simple system of linear equations uniquely determine the moment of the next status change  $T_{k+1}$ , and therefore all the values  $X_{k+1}$ . Thus, we proved point (c) of Theorem 1. We call this transformation of  $X_k$  into  $X_{k+1}$  mapping  $G$ . Note that this mapping does not depend on  $k$ , which makes it easily programmable.

The following example illustrates the dynamic evolution of the system.

**Example 3.** Let  $n = 3$ , matrix  $Q$  be given by (5), and the same debt vector  $\mathbf{b} = (13, 22, 20)$  as in Example 2, but with cash vector  $\mathbf{c} = (\frac{1}{2}, \frac{1}{2}, 0)$ . For this vector  $\mathbf{b}$ , formula (3) yields that  $\mathbf{c}^m = (\frac{2}{3}, \frac{1}{2}, -\frac{7}{6})$ , i.e.,  $\mathbf{c} \prec \mathbf{c}^m$ , so this cash vector makes the model deficient. Formally, this is not a regular case, since  $J_0(0) = \{3\}$ , but this is an example of the Big Bang effect that can be easily fixed because  $n_3 = \frac{1}{2} + \frac{1}{3} > 1$ , and thus bank 3 should be instantly reclassified as positive.

Thus, on the first interval  $\Delta_1 = [T_1, T_2)$ ,  $T_1 = 0$ , we have *all*  $u_i = 1$ . Then  $\mathbf{d} = (-\frac{5}{12}, \frac{1}{4}, \frac{1}{6})$ . Each moment of status change  $T_{i+1} = T_i + t_i, i = 1, 2, \dots$ , is the moment when either some debt is paid up or some cash position hits zero from above. The moment  $t_1 = T_2$  is defined by condition  $c_1(t) = c_1 + d_1 t = 0$ , which yields  $t_1 = \frac{6}{5}$ . Then  $\mathbf{c}(T_2) = (0, \frac{4}{5}, \frac{1}{5})$  and  $\mathbf{b}(T_2) = (b_i - u_i t_1) = (11\frac{4}{5}, 20\frac{4}{5}, 18\frac{4}{5})$ .

On the second interval  $\Delta_2 = [T_2, T_3)$ , we have  $J_+ = \{2, 3\}$  and  $J_0 = \{1\}$ ; thus,  $u_2 = u_3 = 1$  and  $u_1 = n_1 = \frac{7}{12}$ . Then  $\mathbf{d} = (0, \frac{1}{24}, -\frac{1}{24})$ . Moment  $t_2$  is defined by condition  $c_3(t) = \frac{1}{5} + d_3 t = 0$ , and then  $t_2 = 4\frac{4}{5}$ ,  $T_3 = t_1 + t_2$ . Then  $\mathbf{c}(T_3) = (0, \frac{4}{5} + d_2 t_2 = 1, 0)$  and

$$\mathbf{b}(T_3) = (b_i(T_2) - u_i t_2) = (9, 16, 14).$$

On the third interval,  $\Delta_3 = [T_3, T_4)$ , we have  $J_+ = \{2\}$  and  $J_0 = \{1, 2\}$ , which yields  $u_2 = 1$ , and since  $c_1(T_2) = c_3(T_2) = 0$ , we need to solve the equilibrium system (8) with two equations  $u_1 = \frac{1}{3} + \frac{1}{4}u_3$ ,  $u_3 = \frac{2}{3} + \frac{1}{2}u_1$  to find  $u_1 = n_1$  and  $u_3 = n_3$ . This system has a solution  $u_1 = \frac{4}{7}$ ,  $u_3 = \frac{20}{21}$ . Then  $n_2 = \frac{4}{7} \times \frac{1}{2} + \frac{20}{21} \times \frac{3}{4} = 1$  and  $\mathbf{d} = (0, 0, 0)$ . Thus,  $\mathbf{c}(t) = \mathbf{c}(T_3) = (0, 1, 0)$  for  $t \geq T_3$  and moment  $t_3$  is defined by the moment when one of the *debts*  $(9 - \frac{4}{7}t_3, 16 - t_3, 14 - \frac{20}{21}t_3)$  hits zero. Then  $t_3 = 14\frac{7}{10}$  with  $b_3(T_4) = 0$ . At moment  $T_4 = t_1 + t_2 + t_3$ , we have  $\mathbf{b}(T_4) = (\frac{3}{5}, 1\frac{3}{10}, 0)$  and  $\mathbf{c}(T_4) = (0, 1, 0)$ .

On the last interval  $\Delta_4 = [T_4, T_5)$ ,  $J_0 = \{1\}$ ,  $J_+ = \{2\}$ , and  $J_* = \{3\}$ , and thus  $u_1 = n_1 = \frac{1}{3}$ ,  $u_2 = 1$ , and  $u_3 = 0$ . Then  $d = (0, -\frac{5}{6}, \frac{5}{6})$ . Moment  $t_4$  is defined on this interval by condition  $c_2(t) = 1 + d_2 t = 0$ , and then  $t_4 = 1\frac{1}{5}$ . At moment  $T_5 = T_* = t_1 + t_2 + t_3 + t_4$ , the flow stops because bank 2 was the last positive bank. Then cash vector  $\mathbf{c}(T_*) = (0, 0, 1)$  and  $\mathbf{b}(T_*) = (b_i(T_4) - u_i t_4) = (\frac{3}{5} - \frac{1}{3} \times \frac{6}{5}, \frac{13}{10} - \frac{6}{5}, 0) = (\frac{1}{5}, \frac{1}{10}, 0)$ , with banks 1 and 2 in default.

## 5 “Swamps” and Multiplicity of Solutions

In the regular case (all  $c_i > 0$  and  $u_i = 1$  for the positive group), there is a unique solution to the dynamical system, as described in Theorem 1. In this section, we will formulate the results for the general case when there are banks with an initial cash position of zero or even negative. If banks from the positive group have some liabilities to these banks, then after a possible reclassification, all remaining banks in the zero group have positive in-rates, obtained as a solution of system (8), less than one, and therefore they can be treated as zero group in the regular case, and nothing new happens. The new interesting case is when there are banks in the zero group such that banks with positive cash have no obligations, direct or indirect, to them. This case might look exceptional, yet it is critical to the understanding of potential multiplicity of clearing vectors. As we will see, the main reason for multiplicity is that some debts can be settled between banks that do not have any money.

Informally, a *swamp* is a set where all its members at  $t = 0$  have no positive cash, no direct or indirect inflow from the positive group, and have debts but only within this set.

As a result, they do not participate in the basic flow, but they can “run money in a circle”, partially paying their debts while at the same time keeping constant their nonpositive cash positions. Such transactions can be considered to be partial debts restructuring, i.e., a transformation of the initial debt vector into a new one. We will give a formal definition of a swamp in Subsection 5.3.

## 5.1 The Multiplicity Theorem

We start by using the matrix of initial obligations  $Q = Q(0)$  to divide all banks that are not in the paid-up group  $J_*(0)$  into two disjoint sets. Denote  $A_+$  the set of all banks that at moment  $t = 0$  are either positive, have sponsors from the positive group, or have sponsors from the previous group, etc., i.e., have positive direct or indirect sponsors. Formally,  $A_+ = J_+(0) \cup J_{+,1} \cup J_{+,2} \dots$ , where  $J_{+,1}$  is a set of zero banks that have sponsors from  $J_+(0)$ ,  $J_{+,2}$  is a set of zero banks that have sponsors from  $J_{+,1}$ , etc. We denote all other banks that are not in the paid-up group as  $A_0$ . These are banks that have no positive cash and no positive input from the outside at the initial moment and hence forever. They may have liabilities to banks in  $A_+$  and  $J_*(0)$  but not vice versa. The dynamics of a flow in set  $A_+$ , aside from the possible reclassification of some zero banks at  $t = 0$  due to the “Big Bang” effect, which we discuss in Subsection A2, is unique and is described in Theorem 1. The evolution of group  $A_0$  is the main subject of Theorem 2.

Importantly, swamps may exist at the initial moment but they cannot appear later (Lemma 5.3). In a model where the assumption of “no debt seniority” might be modified, swamps could appear in later stages as well. Of course, they do appear in reality, as we mentioned in the Introduction — for example, in the case of *Enron* and in the more recent debacle with *Wirecard* in Germany.

Now we can formulate Theorem 2, which extends the results of Theorem 1 to general flows. Let us represent set  $A_0$  as a union of disjoint sets  $A_0 = U_0 + S_0$ , where  $U_0$  is the set of all transient banks in  $A_0$  and  $S_0$  is the union of all ergodic subclasses in  $A_0$ . Formal definitions are given in the next Subsection 5.2; for now, it is sufficient to say that banks from  $S_0$  will be the ones that can pay each other without any cash on hand.

**Theorem 2.** (a) For any initial cash vector  $\mathbf{c}$ , there exists a basic flow solution of (6),  $\mathbf{p}^*$ , with  $p_i = 0$  for  $i \in A_0$ .

(b) If set  $S_0$  in the decomposition of  $A_0$  is empty, then the basic flow solution is a unique solution of (6), and all banks in  $A_0$  are nonactive.

(c) If set  $S_0 \neq \emptyset$ , i.e., if there is at least one swamp, then there are multiple solutions  $\mathbf{p}'^*$  of the form  $\mathbf{p}'^* = \mathbf{p}^* + \sum_k s_k \mathbf{p}_{*k}$ , where  $\mathbf{p}$  is the basic flow solution,  $0 \leq s_k \leq 1$ , and each vector  $\mathbf{p}_{*k}$  is obtained through an invariant distribution  $\pi_{*k}$  for the corresponding ergodic subclass.

The existence of swamps makes it possible that the flow in the swamp(s) will last longer than  $T_*$ , which is defined as the moment when the last *positive* bank repays its debt. Note that our basic flow solution corresponds to the *smallest*, and the solution with all  $s_k = 1$  in Theorem 2 to the *greatest*, clearing payment vector in Eisenberg and Noe (2001) and Kabanov, Mokbel and El Bitar (2018). They coincide if there are no swamps.

## 5.2 Markov Chains, Basic Flow Evolution, and Invariant Rates

The analysis of “swamps” and the proof of Theorem 2 require introduction of new machinery in the studies of financial clearing, that of Markov chains.

So far, we have not used any stochastic interpretations in the analysis of our deterministic model. Still, it is well known that many statements in probability theory have their deterministic analogs and vice versa. The invariant distribution for a Markov chain was in fact introduced by Gustav Kirchhoff for electrical networks long before the concept of a Markov chain was even formulated.

Kemeny and Snell (1976) is a classic treatise on finite Markov chains; we refer to Shiryaev (2019) as an up-to-date introduction to the subject. If Markov chain  $Z = (Z(n))$  has a state space  $S$  and a transition (stochastic) matrix  $Q$ , and  $\mathbf{m}_k = (m_k(i), i \in S)$  is a vector of the probability distribution of  $Z$  at some moment  $k$ , then  $\mathbf{m}_{k+1} = Q^T \mathbf{m}_k$  is the distribution vector at the next moment. Given any stochastic matrix  $Q$ , all states can be divided into *transient* states and *ergodic* states. Each ergodic state is a member of some ergodic subclass. All transient states form a transient set. A Markov chain eventually leaves transient states moving to one of these subclasses to stay there forever. If subset  $B$  is such an ergodic subclass, the Markov chain can go from any state to any other state (not necessarily in one move). Some textbooks call such subsets *irreducible*, and call them ergodic if they are also aperiodic. In both cases, there is an invariant distribution  $\pi$

that satisfies the equality  $\pi = Q_B^T \pi$ . In the latter case, this distribution is also the limit distribution, and in particular  $Q_B^n \rightarrow A$ , where  $A$  is a stochastic matrix with identical rows equal to  $\pi$ .

The evolution of the stochastic matrix  $Q(t)$  that starts with matrix  $Q$  at moment  $t = 0$  reflects the evolution of the partition  $P(t)$  as follows. If bank  $i$  is paid-up, we consider  $i$  to be an absorbing state, i.e.,  $q_{ii}(t) = 1$ . For our purposes, it is more convenient to distinguish between the absorbing and ergodic states, assuming that every ergodic subclass has at least two states.

Obviously, matrices  $Q(t)$  are constants on the status intervals and change only at moments  $T_k$ . The set of absorbing states at moment  $t$  is just  $J_*(t)$ , but banks in the transient set and in the ergodic subclasses may have a positive or zero status. Thus, each bank at each moment has a dual classification: by status and by state.

We need to formulate a few facts about transient sets and ergodic subclasses. Suppose that  $B \subseteq J$  and the states in  $B$  are transient. Then  $Q_B$  is a substochastic matrix, i.e., the sum of coefficients in some rows is less than one, and the Markov chain leaves set  $B$  at some moment with probability 1 or, equivalently, matrix  $Q_B^n$  tends to zero when  $n$  approaches infinity. Thus, the matrix  $N_B = (I - Q_B)^{-1}$  is well-defined. Matrix  $N_B$  is called the *fundamental* matrix for substochastic matrix  $Q_B$ , and has a natural probabilistic interpretation. Namely,  $n_B(x, y)$  equals the expected number of visits to state  $y$  for a Markov chain with the initial point in  $x$  before the exit out of set  $B$ . We also have  $N_B = \sum_{n=0}^{\infty} Q_B^n$  and  $N_B^T = (I - Q_B^T)^{-1}$ .

Let  $B = J_0(t)$  for some time  $t$ , and  $Q_B$  be the corresponding (sub)stochastic matrix. Let us introduce vector  $\mathbf{e}_B$ , the “input vector” from “sponsors” of the positive group with their out-rates  $u_i = 1$  to zero group  $B$ . The coordinates of  $\mathbf{e}_B$  are defined by  $e_{i,B} = \sum_{j \in J_+} q_{ji}, i \in B$ . The out-rates for the zero group  $B$ , which are equal to their in-rates, are denoted by  $v_B \equiv v$ . According to (C2), these rates, defined by formula (8), satisfy the equation

$$\mathbf{v}_B = \mathbf{e}_B + Q_B^T \mathbf{v}_B. \tag{12}$$

The question is, when does this equation have a solution? The answer is given by the following lemma.

**Lemma 5.1.** (a) *If  $Q_B$  is a substochastic matrix, then the solution of (12) is given by*

formula

$$\mathbf{v}_B = (I - Q_B^T)^{-1} \mathbf{e}_B \equiv N_B^T \mathbf{e}_B. \quad (13)$$

(b) If  $Q_B$  is a stochastic matrix and  $\mathbf{e}_B \neq 0$ , then equation (12) has no solution; if  $\mathbf{e}_B = 0$ , then all solutions of (12) are proportional to the solution given by the invariant distribution for the corresponding Markov chain,  $\pi = Q_B^T \pi$ .

(c) The solutions  $\mathbf{v}$  and  $\mathbf{v}'$  of the equations  $\mathbf{v} = \mathbf{e} + Q_B^T \mathbf{v}$  and  $\mathbf{v}' = \mathbf{e}' + Q_B^T \mathbf{v}'$  satisfy  $\mathbf{v} \leq \mathbf{v}'$  if  $\mathbf{e} \leq \mathbf{e}'$ .

The proof of points (a) and (b) of Lemma 5.1 follows from simple facts from linear algebra. Point (c) follows immediately from point (a), since the elements of matrix  $N_B^T$  are nonnegative.

Before we proceed to the next step, we need to describe how invariant out-rates can be used. Suppose that a subset of banks  $B$ , say at moment 0, forms an ergodic subclass with an invariant distribution  $\pi$ . Recall that all banks are partitioned into sets  $A_+$  and  $A_0$  (see Subsection 5.1). If set  $B \subseteq A_+$ , then it may have positive and zero group members; if set  $B \subseteq A_0$ , then it has only zero group banks. In both cases we can use the invariant probabilities  $\pi$  as out-rates  $u_i$ . By the definition of  $\pi$ , this guarantees that the out-rates are equal to the in-rates for each bank in  $B$ . This implies that all  $c_i(t)$  remain equal to the initial values  $c_i$  until the moment when these out-rates are no longer in effect. At the same time, each  $b_i(t) = b_i - tu_i$  decreases at a positive rate  $u_i$ . Let  $t_i = \max_t \{t : b_i(t) > 0\}$ ,  $T = \min_i t_i$ , and  $i_1$  be the bank where this minimum is achieved. As usual, we assume that there is only one such bank, say  $j$ . It means that  $b_j(T) = 0$ , this bank has no liability to any bank in  $B$ , and since  $B$  is a closed subset, no liability to any bank in  $J$ , i.e., its status becomes paid-off. For all other banks in  $B$ ,  $b_i(T) = b_i - Tu_i = b_i^{(1)} > 0$ . Thus, we restructured the debts and our model without changing the cash positions of any bank in  $B$ .

Note that at moment  $T$  all banks in  $B \setminus j$  become transient, because some of them definitely have obligations to bank  $j$ , and because before this moment, by definition of an ergodic set, they were connected to bank  $j$ . Therefore, they will remain transient or become absorbing. We denote the corresponding payment vector obtained in this process as  $p_B^*$ . Note that we can change the out-rates  $u_i = \pi_i$  to some proportional  $s\pi_i$  with some coefficient  $s > 0$ . Since  $\max_i \pi_i = q < 1$ , we can select any  $0 < s < 1/q$  without a violation

of our assumption (C1) that guarantees that out-rates cannot exceed 1. Then the time of restructuring  $T$  will be also changed proportionally to  $sT$ , but the payment vector will be the same because the rate of payment will be increased (decreased) in proportion to the decreased (increased) period of payment; see equations (10). Of course, if payments are stopped before one of the debts hits zero, the clearing vector will be smaller.

The important point about this construction is that the cash position of each bank in set  $B$  was not changed, so it is possible that some of these banks did not have cash at all or even had negative positions. In particular, we obtained the following result.

**Lemma 5.2.** *Let set  $B$  be an ergodic subclass, and all  $c_i \leq 0$ ,  $i \in B$ . All nontrivial solutions of the clearing equation (6) that are limited to this set can be obtained using only invariant out-rates. All these solutions are proportional to vector  $\mathbf{p}_B^*$ , with the coefficient of proportionality not exceeding 1.*

We illustrate this construction by a simple example.

**Example 4.** Let  $n = 3$ ,  $\mathbf{b} = (2, 3, 4)$ ,  $c_i \leq 0$ ,  $i = 1, 2, 3$ , and matrix  $Q$  be given by (5). It is easy to check that the invariant distribution for the corresponding Markov chain is  $\pi = \frac{1}{53} (12, 21, 20)$ . Let us select these fractions as the out-rates  $u_i$  for these three banks. By definition of  $\pi$ , we have  $u_i = n_i$ ,  $i = 1, 2, 3$ , and therefore  $c_i(t) = c_i$ ,  $i = 1, 2, 3$  on some interval  $\Delta = [0, t)$ . On this interval we have  $b_i(t) = b_i - u_i t$ ,  $i = 1, 2, 3$ . Then at moment  $T = \frac{53}{7}$ , we obtain that  $b_2(T) = b_2 - u_2 T = 3 - \frac{21}{53} \times \frac{53}{7} = 0$ , and  $b_1(T) = b_1 - u_1 T = 2 - \frac{12}{53} \times \frac{53}{7} = \frac{2}{7} > 0$ ,  $b_3(T) = b_3 - u_3 T = 4 - \frac{20}{53} \times \frac{53}{7} = \frac{8}{7} > 0$ . We have  $J_*(T) = \{2\}$ ,  $J(T) = \{1, 3\}$  and since  $c_i \leq 0$  for  $i = 1, 3$  we obtained that  $T = T_*$ , the vector of unpaid debts is  $(\frac{2}{7}, 0, \frac{8}{7})$ , and the clearing vector  $\mathbf{p}^* = p(T) = \mathbf{b} - \mathbf{b}(\mathbf{T}) = (\frac{12}{7}, 3, \frac{20}{7})$ . The remaining debts of banks 1 and 3 to bank 2,  $\frac{1}{7}$  and  $\frac{6}{7}$ , respectively, are not repayable, but their *mutual remaining debts* ( $\frac{1}{7}$  and  $\frac{2}{7}$ , respectively) can be cancelled only if we allow violating the proportionality assumption (A). Then, bank 2 owes nothing to bank 3, and bank 3 owes  $\frac{1}{7}$  to bank 2. The model cannot be restructured any further.

### 5.3 Swamps

The situation when all banks in ergodic set  $B$  have zero or negative cash gives rise to an important notion of swamps that in the general case lead to non-uniqueness of the solution

of the clearing equation. Formally, we say that *swamp*  $B$  is an ergodic subclass of banks that have zero or negative cash at  $t = 0$ . The next Lemma 5.3 describes the evolution of the banks from the point of view of their positions as states of Markov chains. The irreversibility of a status change is accompanied by the irreversibility of a state change. (As above, we assume that only one bank at a time may change its status.) Part (c) establishes formally that swamps might exist at the initial moment but cannot appear later. Recall that  $A_+$  are the banks that participate in basic flow (regular case), and  $A_0$  are the banks that have no positive cash and no positive input from the outside at the initial moment and hence forever. They may have liabilities to banks in  $A_+$  and  $J_*(0)$ , but not vice versa.

**Lemma 5.3.** *The only possible evolution of banks as states of the Markov chain is the following.*

- (a) *Paid-up banks are always absorbing, and transition to this state is irreversible;*
- (b) *Positive banks, as well as zero banks in  $A_+$ , may be ergodic or transient; as ergodic, they can become transient or absorbing, and as transient, they can only become absorbing;*
- (c) *Zero banks in  $A_0$  may be transient or ergodic; if transient, they are always transient and inactive, i.e., for them,  $u_i(t) \equiv 0$  for all  $t$ ; if there is an ergodic subclass, then its members can be active only under invariant out-rates until the moment when one of them becomes absorbing, and then all other members become transient and inactive. Thus, swamps can exist from the beginning but cannot appear later.*

*Proof.* Part (a) is straightforward. Positive banks may be ergodic or transient. With a status of an ergodic or a transient bank changed to paid-up, its state becomes absorbing. All other states in the ergodic subset become transient because now they are connected to an absorbing state. If a zero bank in  $A_+$  is transient, its out-rate is positive and hence it may become an absorbing state or remain transient until the end, and it means that this bank ends up in default. This proves (b).

If  $B$  is the set of all transient banks from  $A_0$ , then the input vector  $e_B = 0$  for all moments  $t$ , and by part (a) of Lemma 5.1, the only solution for the equilibrium rates is zero, so they remain nonactive forever.

If  $B$  is an ergodic subclass in  $A_0$ , then the only possible out-rates  $a_i, i \in B$ , by Lemma 5.2, must be proportional to the invariant out-rates  $\pi_i$ , where  $\pi_i$  is the invariant

distribution for  $Q_B$ . Only these out-rates can keep  $c_i(t) \equiv c_i$  on some interval. When the invariant out-rates  $\pi_i$  are applied, then on some interval  $[0, T)$ , all debts will be decreased to the level  $b_i - \pi_i T \geq 0$  with exactly one bank  $j$  with  $b_j(T) = 0$ , and all other banks with  $b_i(T) > 0$ . At moment  $T$ , bank  $j$  changes its status to paid-up, and its state becomes absorbing. Then all other banks in  $B$  become transient and become nonactive. Thus, all transient banks in  $A_0$  will never be able to repay their debts, but the banks in each of the ergodic subclasses will be able to restructure and reduce their debts to each other. This proves (c).  $\square$

Now we have everything to complete the proof of Theorem 2.

**Proof of Theorem 2.** Part (a) follows from Theorem 1 and the fact that  $p_i = 0$  for  $i \in A_0$  obviously satisfies the clearing equation (6).

Part (b) follows from part (a) of Lemma 5.3.

Part (c) follows from Lemmas 5.3 and 5.2, and the fact that if the number  $k$  of the irreducible subclasses is greater than one, they have an empty intersection, and therefore any linear combination of these solutions with coefficients  $0 \leq s_k \leq 1$  is also a solution of (6) with coordinates  $p_{*i} \leq b_i$  for  $i \in B$  and  $p_{*i} = 0$  for  $i \notin B$ .  $\square$

## 6 The Cascade Theorem and the Cascade Algorithm

In introducing our model, we asked the following question: What is the smallest initial cash vector  $\mathbf{c}$  such that *all banks* ultimately pay their debts? The answer was vector  $\mathbf{c}^m$  satisfying (3). The next natural question is as follows. Given matrix  $Q$  and vector  $\mathbf{b}$ , what is the smallest vector  $\mathbf{c}$  such that *at least one bank* will pay its debt? The answer to this question might seem surprising: a bank that fully repays its debt exists for *any* initial cash vector  $\mathbf{c} = (c_1, \dots, c_n)$  if at least one  $c_i > 0$ .

In this section, we will show that any *sufficient* model  $M = (\mathbf{b}, Q, \mathbf{c})$  can be restructured into an equivalent cascade model in which there is a group of banks that can pay off their debts even with negative cash positions, a group of banks that need to pay only to the banks in the first group, and so on.

## 6.1 The Cascade Theorem

Given any model  $M = (\mathbf{b}, Q, \mathbf{c})$ , we say that payments  $\mathbf{p} = (p_1, \dots, p_n)$  are *admissible* if they do not exceed the obligations of the banks and are proportional to their obligations. We say that model  $M' = (\mathbf{b}', Q', \mathbf{c}')$  is *obtained from model*  $M = (\mathbf{b}, Q, \mathbf{c})$  *by restructuring* if the second model is obtained from the first via admissible payments.

Note that we do not assume that these proportions are equal. Under admissible payments, one bank can pay 10% of its debt, and the other 20%. Also, observe that the clearing vector  $p^*$  in the initial model is in fact the maximum possible vector of admissible payments, i.e.,  $p^* \succ p$  for any admissible  $p$ .

This definition implies in particular that  $b_{ij} \geq b'_{ij}$  for all  $i, j$ . We also say that  $M$  is *reduced* to  $M'$ , and denote  $M \rightarrow M'$ . It is easy to see that if  $(p^*)'$  is the clearing vector in this new model, then the clearing vector in the initial model  $p^* = p + (p^*)'$ . We also may consider a sequence of restructured models  $M \rightarrow M_1 \rightarrow M_2 \dots \rightarrow M_N$  to obtain a schedule of payments.

We say that a model has a *cascade* structure, or simply that a model is a cascade, if set  $J$  can be partitioned into sets  $(J_1, J_2, \dots, J_m)$  such that the structure of obligations has the following form: there is a group of banks  $J_1$  without any debts, there is a group of banks  $J_2$  that owe money only to banks in  $J_1$ , etc., and, finally, there is a group of banks  $J_m, m \leq n$ , to which no bank owes money, and these banks may have debts to any of the other banks.

The structure of a cascade shows which banks are the main sources of debt, and where the weakest point is in this chain of payments. The assumption that the initial model is sufficient is important. Otherwise, the process of mutual cancellations of debts inside of some subgroup of banks, that lead to a restructuring into a cascade, is impossible because of the strong assumption (A). Still, even in a deficient case, the cascade can be used to find where the infusion of cash can alleviate the burden of obligations (see more in [Farthing and Sonin, 2021](#)). We provide a fully worked-out example of a cascade restructuring in Subsection 6.2, using equations (10) and (11) that determine the discrete timing of status changes.

**Theorem 3.** (a) *Given any model  $(\mathbf{b}, Q, \mathbf{c})$ , there is at least one bank  $j$  that is able to pay its debt without any money, i.e., having  $p_j = b_j$ , even with  $c_j \leq 0$ ;*

(b) Given any sufficient model  $(\mathbf{b}, Q, \mathbf{c})$ , it can be restructured into an equivalent cascade model  $(\mathbf{b}', Q, \mathbf{c}')$ .

The proof of Theorem 3 is given by the Cascade Algorithm, based on the explicit construction presented in the next subsection.

## 6.2 The Cascade Algorithm

We provide a formal description of the algorithm in the proof of Theorem 3. Example 5 demonstrates the basic logic of the proof.

**Proof of Theorem 3.** If at time 0 there is at least one paid-off bank, (a) is trivially true. Otherwise, there is an ergodic subclass  $B$  with an invariant distribution  $\pi_i > 0, i \in B$ . Then by Lemma 5.2, applying invariant rates  $u_i = \pi_i$  to this subclass, we guarantee that all  $c_i(t)$  remain equal to the initial values  $c_i$ . We also have that all  $b_i(t) = b_i - tu_i$  are strictly decreasing. Let  $t_i = \max_s \{s : b_i(s) > 0\}$ ,  $T_1 = \min_i t_i$ , and  $i_1$  be the bank where this minimum is achieved. For simplicity, we assume that there is only one bank with this property; otherwise, bank  $i_1$  will be replaced by subset  $J_1$ . It means that  $b_{i_1}(T_1) = 0$ , and this bank has no liabilities to any of the banks in  $B$ , and therefore to any of the banks in  $J$ . All other banks in  $B$  have debts  $b_i(T_1) = b_i - T_1 u_i = b_i^1 > 0$  to banks in  $B$  only.

Let us denote the set of remaining banks  $B \setminus i_1 = S^1$  and represent  $n - 1$  dimensional vector of the remaining debts of these banks,  $\mathbf{b}^{(1)} = \mathbf{b}(T_1)$ , as a sum of two vectors  $\mathbf{b}^{(1)} = \mathbf{f}^{(1)} + \mathbf{e}^{(1)}$ , where each component  $f_i^{(1)}$  is the remaining debt of bank  $i$  to bank  $i_1$ , i.e.,  $f_i^{(1)} = b_i(T_1)q_{ii_1}$ ,  $i \in S^1$ . Then vector  $\mathbf{e}^{(1)}$  is an  $(n - 1)$ -dimensional vector of the remaining liabilities of banks in  $S^1$  to each other. Let us obtain an  $(n - 1)$ -dimensional stochastic matrix  $Q^{(1)} = \{q_{ij}^{(1)}\}$ ,  $i, j \in S^1$  by removing in  $Q$  row and column  $i_1$ , i.e., all entries  $q_{i_1 i}$  and  $q_{i i_1}$ ,  $i \neq i_1$ , and normalizing the new rows, i.e.,  $q_{ij}^{(1)} = q_{ij} / \sum_{j \neq i_1} q_{ij}$ . If there is bank  $i$  or a few banks in  $S^1$  such that  $q_{ii_1} = 1$ , they should be excluded from set  $S^1$  and will be included later in set  $J_2$ . After such a modification of set  $S^1$ , all remaining banks form an ergodic subclass. Indeed, all banks left have a connection with some bank in this group.

The next step is to obtain the invariant distribution  $(\pi_i^1), i \in S^1$  for matrix  $Q^{(1)}$ . Then by Lemma 5.2, applying invariant rates for this subclass, at some moment  $T_2$  we obtain

bank  $i_2$  that paid its debts without changing its cash position. In other words, we apply the same step to the set of banks  $S^1$  with matrix  $Q^{(1)}$  and the debt vector  $\mathbf{e}^{(1)}$  as we applied initially to set  $J$  with matrix  $Q$  and the debt vector  $\mathbf{b}$ . If at each moment  $T_i$  only one bank pays its debt, then obviously in  $n - 1$  steps we obtain a cascade with one bank at each level. Otherwise, in a smaller number of steps we obtain the sets mentioned in Theorem 3. The sequence of payments is given by vectors  $\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \dots, \mathbf{f}^{(n-1)}$ .  $\square$

**Example 5.** Let  $n = 4$  and matrix  $Q$  be given by  $q_{12} = q_{13} = q_{14} = \frac{1}{3}$ ,  $q_{21} = \frac{1}{4}$ ,  $q_{23} = \frac{1}{2}$ ,  $q_{24} = \frac{1}{4}$ ,  $q_{31} = \frac{1}{6}$ ,  $q_{32} = \frac{1}{2}$ ,  $q_{34} = \frac{1}{3}$ ,  $q_{41} = \frac{1}{6}$ ,  $q_{42} = \frac{1}{2}$ , and  $q_{43} = \frac{1}{3}$ . The invariant distribution for the Markov chain defined by  $Q$  is  $\pi = \frac{1}{508}(84, 160, 147, 117)$ . Consider the debt vector  $\mathbf{b} = (33\frac{67}{78}, 56\frac{80}{117}, 55\frac{5}{39}, 20)$ . Using formula (3), we obtain  $\mathbf{c}^m = (7\frac{1}{6}, 7\frac{5}{6}, 8\frac{5}{6}, -23\frac{5}{6})$ .

$$Q = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/4 & 0 & 1/2 & 1/4 \\ 1/6 & 1/2 & 0 & 1/3 \\ 1/6 & 1/2 & 1/3 & 0 \end{pmatrix} \quad (14)$$

Suppose that cash vector  $\mathbf{c} \succ \mathbf{c}^m$ , so our model is *sufficient*. Our goal is to get to an equivalent cascade model. As in Example 3 and Lemma 5.2, on the first interval we are going to use  $u_i = \pi_i, i = 1, \dots, 4$ . Then on this interval  $c_i(t) = c_i$  for all  $t$ , and hence the moment  $T_1 = t_1$  is the first time when one of  $b_i(t)$  hits zero. It is easy to check that this is bank  $i = 4$  and  $t_1 = 20/\pi_4$ . For  $i = 1, 2, 3$ , we have  $b_i(t_1) = b_i - 20 \times \pi_i/\pi_4$ :  $b_1(t_1) = b_1 - 20 \times \frac{84}{117} = 19\frac{1}{2}$ ,  $b_2(t_1) = b_2 - 20 \times \frac{160}{117} = 29\frac{1}{3}$ ,  $b_3(t_1) = b_3 - 20 \times \frac{147}{117} = 30$ ,  $b_4(t_1) = b_4 - 20 = 0$ .

Let us represent the vector of the remaining debts of all banks at time  $t_1$  as the sum of two vectors,  $\mathbf{b}(t_1) = \mathbf{f}^{(1)} + \mathbf{e}^{(1)}$ , where vector  $\mathbf{f}^{(1)}$  represents the vector of the remaining debts of banks 1, 2, 3 to bank 4, and vector  $\mathbf{e}^{(1)}$  represents the vector of the remaining debts of banks 1, 2, 3 to each other. We find these vectors using matrix  $Q$ . Thus,  $f_i^{(1)} = b_i(t_1)q_{i4}$ , and  $e_i^{(1)} = b_i(t_1) - f_i^{(1)}$ . We obtain  $\mathbf{f}^{(1)} = (19\frac{1}{2} \times \frac{1}{3} = \frac{13}{22}, 29\frac{1}{3} \times \frac{1}{4} = \frac{22}{3}, 30 \times \frac{1}{3} = 10) = (6\frac{1}{2}, \frac{22}{3}, 10)$  and  $\mathbf{e}^{(1)} = (13, 22, 20)$ . This is the debt vector from Example 2. So, we may consider a sub-model with only three banks, 1, 2, and 3, and transition matrix  $Q^{(1)}$  obtained from  $Q$  by removing row 4 and column 4, and re-normalizing the new rows.

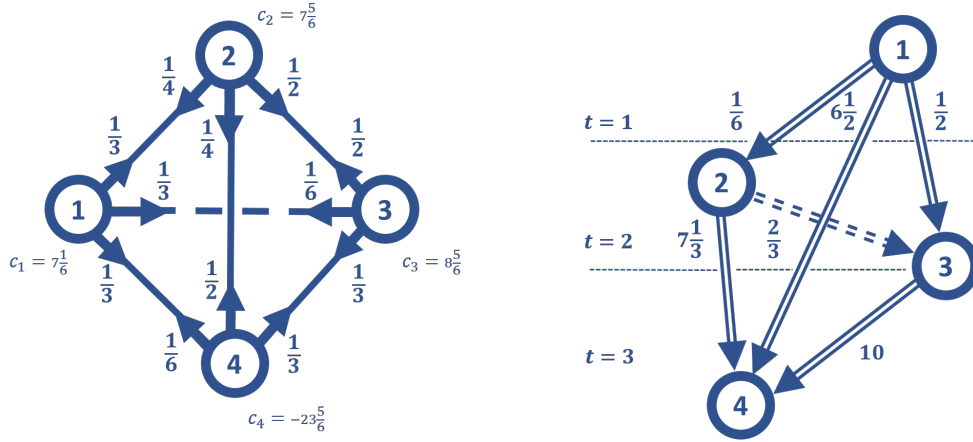


Figure 2: (a) The initial matrix of relative liabilities with the minimum cash vector as the initial cash vector  $\mathbf{c}$ . (b) The cascade of payments in three steps.

(Matrix  $Q^{(1)}$  was analyzed in Example 2.)

The situation with debt vector  $\mathbf{b} = \mathbf{e}^{(1)} = (13, 22, 20)$  was analyzed in Example 2. Using vector  $\mathbf{e}^{(1)} = (\frac{13}{2}, \frac{22}{3}, 10)$  and cascade 1(b) from Example 2, we obtain the following cascade. At step 1, bank 1, having minimum cash of  $7\frac{1}{6}$ , sends a check for  $6\frac{1}{2}$  to bank 4, for  $\frac{1}{6}$  to bank 2, and for  $\frac{1}{2}$  to bank 3. Note that  $7\frac{1}{6} - (6\frac{1}{2} + \frac{1}{6} + \frac{1}{2}) = 0$  because  $c^*_1 = 0$ . At step 2, bank 2, having minimum cash of  $7\frac{5}{6}$ , and  $\frac{1}{6}$  received from bank 1, sends a check for  $\frac{2}{3}$  to bank 3, and  $7\frac{1}{3}$  to bank 4. Here we have  $7\frac{5}{6} + \frac{1}{6} - (\frac{2}{3} + 7\frac{1}{3}) = 0$ . At step 3, bank 3, having minimum cash of  $8\frac{5}{6}$ , with  $\frac{1}{2}$  received from bank 1, and  $\frac{2}{3}$  received from bank 2, sends a check for 10 to bank 4. Here we have  $8\frac{5}{6} + \frac{1}{2} + \frac{2}{3} - 10 = 0$ . Bank 4, having minimum cash of  $-23\frac{5}{6}$ , received in total  $6\frac{1}{2} + 7\frac{2}{6} + 10 = 23\frac{5}{6}$ . Figure 2(b) depicts the payments in this cascade. As always the case with a sufficient model, the payments can be uniquely identified using the cascade and vectors  $\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \dots, \mathbf{f}^{(n-1)}$ . All banks paid their debts and their final cash positions are 0's if  $\mathbf{c} = \mathbf{c}^m$ . If  $\mathbf{c} \succ \mathbf{c}^m$ , then it might be possible to settle all debts in less than three steps.

## 7 Extensions

While our setup uses the initial setting of Eisenberg and Noe (2001) as the starting point, our model can be naturally generalized to work with many extensions. For example, a modification of our algorithm can work with liabilities or shares of different seniorities or

liabilities with multiple maturities. In this section we briefly discuss possible extensions.

## 7.1 Debt Seniority

First, we can assume that matrix  $Q$  is not obtained by the equalities  $q_{ij} = b_{ij}/b_i$ , and just represents the priority of payments. We will need the following modifications. Now all out-pipes from tank  $i$  are not closed simultaneously, but one by one, when the corresponding debt is paid, i.e.,  $(i, j)$  pipe is closed when debt  $b_{ij}$  is paid. The corresponding equations and the times of status change can be easily modified. If we assume that some debts should be paid not just faster but before other payments, then at the initial moment not all  $(i, j)$  pipes are open but only for  $j$  in the senior (for  $i$ ) class. When these debts are paid, the other group of  $(i, j)$  pipes is open, etc.

Similarly, while we assumed that all “positive” banks have the same total out-capacities equal to one, the analysis is easily extended to heterogeneous rates (if the regulator considers it important to prioritize some payments). Note that although the dynamics of the continuous-time model will be changed, the clearing vector will be the same if the proportionality of all payments is the same. Of course, if the requirement of proportionality of payments is changed, then not only will the dynamics of the continuous-time model be changed but the clearing vector as well. Because of the irreversibility property, not all banks can leave group  $J_+(0)$  before  $T_*$ ; we can have a raw estimate  $T_* \leq \max_i b_i$ .

Let us briefly discuss the following potential question: *how much extra money should be injected in the system to avoid default of a certain group of banks?* It is possible that the sum of extra money is substantially smaller than the total amount of unpaid debts. As we demonstrated, to avoid any defaults, each bank should have  $c_i \geq c_i^m$ . The answer for a group of banks is more nuanced. Without going into much detail, we state that the answer depends on the final debt vector  $\mathbf{b}^* = \mathbf{b}(T_*)$  and the final transient matrix  $Q^* = Q(T_*)$ .

## 7.2 Multiple Maturities

A recent paper, [Kusnetsov and Veraart \(2019\)](#), analyzes the case of multiple maturities. Our model allows a straightforward extension to incorporate this possibility. Suppose that there is a scale of time  $[0, +\infty)$ , and each bank may have obligations with possibly different

due dates, i.e., all liabilities have extra maturity time (stamp)  $b_{ij}(t_k), k = 1, 2, \dots, n_i$ . As in [Kusnetsov and Veraart \(2019\)](#), we start with two due dates for all banks, short and long,  $t_1 < t_2$ . We can modify our model as follows. All parameters of the system of banks are “doubled”, and have an extra index 1 or 2, such as  $b_i(s) = \sum_j b_{ij}(s), s = 1, 2$ . All banks have two types of tanks, *basic* (1) and *dormant* (2), two cash positions  $c_i(s), s = 1, 2$ , and two types of outgoing and incoming pipes, also marked as short (1) and long (2). The dormant tank of bank  $i$  represents an account that can be used only starting after the short moment, and which accumulates money from other banks that have short-term liabilities to  $i$ . All “short” pipes are directed to short tanks, and all “long” pipes are directed to long tanks.

To modify the capacities, we will have to introduce two new axioms. The first, similar to [Kusnetsov and Veraart \(2019\)](#), stems from the requirements of the UK insolvency rules: all liabilities with different maturities are treated with the same priority within the same seniority class. It might be reasonable to introduce a second assumption, different from [Kusnetsov and Veraart \(2019\)](#), where “a bank that is liquidated under the insolvency rules ceases to exist and cannot recover even if the liquidators recover sufficient assets to fully compensate all creditors”. Specifically, one can require that all banks that are in “short default” *must* borrow money at a fixed interest from outside banks to meet their short-term obligations if their long-term receivables from other banks will allow them to avoid insolvency in the long run. This would allow to extend our analysis to the environment with multiple maturities. The extension of our model into a system with multiple maturities will, in turn, allow to introduce stochastic factors. The most natural way is to allow cash positions and partly debt structure to be changed randomly following a Markov chain at moments of due dates.

### 7.3 Clearing Equation as the Bellman Optimality Equation

In our continuous-time model, we used the mathematical technique of Markov chains to analyze the multiplicity of solutions in a deterministic dynamical system. In fact, there is a deeper relationship between the Eisenberg and Noe problem and some classic problems in the theory of Markov chains. The clearing equation (1) has strong resemblance to the

nonlinear equation

$$\mathbf{v} = \max(\mathbf{c} + Q\mathbf{v}, \mathbf{b}), \tag{15}$$

used in many applications that use Bellman optimization. Most linear models, e.g., the Leontief closed and open models with extra constraints, will satisfy (15).

Consider the classic problem of *optimal stopping of a Markov chain* with transition matrix  $Q$ , where a decision maker observing the chain has, at each moment in time, two options, either to continue or to stop (Puterman, 2014). In such a setting, the Bellman (optimality) principle takes the form of equation (15), where vector  $v = (v_i)$  is the value function, i.e.,  $v_i$  is the maximum possible expected reward over all possible stopping times if the Markov chain starts at state  $i$ ,  $c = (c_1, \dots, c_n)$  is a vector that consists of *current rewards*  $c_i$  in state  $i$ , and  $b = (b_1, \dots, b_n)$  is a vector of *terminal rewards*, where  $b_i$  is the terminal reward if the Markov chain stops at state  $i$ . Note that in the Bellman equation, the maximum is taken instead of the minimum, and the straight matrix  $Q$  is used instead of the transposed one.

A simple recursive algorithm, the *State Elimination Algorithm*, to solve the Bellman equation (15) was developed in Sonin (1999, 2006) and was modified to calculate the classic and generalized Gittins indices in Sonin (2008). The relationship between the Eisenberg and Noe basic equation (1) and the optimal stopping problem defined by (15), and the use of the State Elimination Algorithm to solve it, is discussed in Kabanov, Mokbel and El Bitar (2018). Presman (2011) modifies the algorithm to consider the optimal stopping problem with an arbitrary state space and continuous time.

## 8 Conclusion

In this paper, we develop a continuous-time model of clearing in financial networks. This approach provides an intuitive and simple recursive solution to a classical static model of financial clearing introduced by Eisenberg and Noe (2001). The same approach provides a useful tool to solve nonlinear equations involving a linear system and max min operations similar to the Bellman equation for the optimal stopping of Markov chains and other optimization problems. Allowing financially constrained banks to repay their debts at the maximum-available speed without any liquidity injections or guarantees will

result in a clearing payment vector. Our results show that, at least theoretically, there is no need in detailed regulation in a situation of financial distress if the mechanism of resolving simultaneous payments is set right. On the other hand, our model provides a convenient tool to study the optimal strategy for a central regulatory agency to minimize the potential contagion effect triggered by failure of some banks. Finally, because our continuous-time model is equivalent to a discrete dynamical system with easily calculated and programmable standard steps, it can be transformed into a real-time scheduling program with multiple maturity dates that takes into account stochastic factors. The idea of transforming the initial network into a more transparent cascade structure may help to alleviate the transparency versus complexity trade-off.

## References

- Acemoglu, Daron, Asuman Ozdaglar and Alireza Tahbaz-Salehi. 2015. “Systemic risk and stability in financial networks.” *American Economic Review* 105(2):564–608.
- Allen, Franklin and Douglas Gale. 2000. “Financial Contagion.” *Journal of Political Economy* 108(1):1–33.
- Alvarez, Fernando and Gadi Barlevy. 2015. “Mandatory Disclosure and Financial Contagion.” *NBER Working Paper* (21328).
- Amini, Hamed, Damir Filipović and Andreea Minca. 2016. “Uniqueness of equilibrium in a payment system with liquidation costs.” *Operations Research Letters* 44(1):1 – 5.
- Amini, Hamed, Rama Cont and Andreea Minca. 2016. “Resilience to Contagion in Financial Networks.” *Mathematical Finance* 26(2):329–365.
- Azizpour, Shahriar, Kay Giesecke and Gustavo Schwenkler. 2018. “Exploring the sources of default clustering.” *Journal of Financial Economics* 129(1):154–183.
- Banerjee, Tathagata, Alex Bernstein and Zachary Feinstein. 2020. “Dynamic Clearing and Contagion in Financial Networks.” *mimeo* .
- Battiston, Stefano, Domenico Delli Gatti, Mauro Gallegati, Bruce Greenwald and Joseph E. Stiglitz. 2012. “Default cascades: When does risk diversification increase stability?” *Journal of Financial Stability* 8(3):138 – 149.
- Bernard, Benjamin, Agostino Capponi and Joseph E Stiglitz. 2017. Bail-ins and Bail-outs: Incentives, Connectivity, and Systemic Stability. Technical Report 23747 National Bureau of Economic Research.
- Capponi, Agostino, Felix C Corell and Joseph E Stiglitz. 2020. “Optimal Bailouts and the Doom Loop with a Financial Network.” *NBER Working Paper* (27074).
- Castiglionesi, Fabio and Mario Eboli. 2018. “Liquidity Flows in Interbank Networks.” *Review of Finance* 22(4):1291–1334.

- Cifuentes, Rodrigo, Gianluigi Ferrucci and Hyun Song Shin. 2005. “Liquidity Risk and Contagion.” *Journal of the European Economic Association* 3(2-3):556–566.
- Csóka, Péter and Jean-Jacques Herings. 2018. “Decentralized Clearing in Financial Networks.” *Management Science* 64(10):4681–4699.
- Eboli, Mario. 2013. “A flow network analysis of direct balance-sheet contagion in financial networks.” *Kiel Institute for the World Economy Technical report 1862* .
- Eboli, Mario. 2019. “A flow network analysis of direct balance-sheet contagion in financial networks.” *Journal of Economic Dynamics and Control* 103:205 – 233.
- Eisenberg, Larry and Thomas H Noe. 2001. “Systemic risk in financial systems.” *Management Science* 47(2):236–249.
- Elliott, Matthew, Benjamin Golub and Matthew O. Jackson. 2014. “Financial Networks and Contagion.” *American Economic Review* 104(10).
- Elsinger, Helmut. 2009. Financial Networks, Cross Holdings, and Limited Liability. Working Papers 156 Oesterreichische Nationalbank (Austrian Central Bank).
- Farthing, Chris and Isaac Sonin. 2021. “Bankruptcy Mitigation in Financial Networks.” *mimeo* .
- Feinstein, Zachary. 2017. “Financial contagion and asset liquidation strategies.” *Operations Research Letters* 45(2):109 – 114.
- Feinstein, Zachary, Weijie Pang, Birgit Rudloff, Eric Schaanning, Stephan Sturm and Mackenzie Wildman. 2018. “Sensitivity of the Eisenberg–Noe Clearing Vector to Individual Interbank Liabilities.” *SIAM Journal on Financial Mathematics* 9(4):1286–1325.
- Freixas, Xavier, Bruno M Parigi and Jean-Charles Rochet. 2000. “Systemic Risk, Interbank Relations, and Liquidity Provision by the Central Bank.” *Journal of Money, Credit and Banking* 32(3):611–638.
- Glasserman, Paul and H. Peyton Young. 2015. “How likely is contagion in financial networks?” *Journal of Banking and Finance* 50:383 – 399.

- Glasserman, Paul and H. Peyton Young. 2016. “Contagion in Financial Networks.” *Journal of Economic Literature* 54(3):779–831.
- Healy, Paul M. and Krishna G. Palepu. 2003. “The Fall of Enron.” *Journal of Economic Perspectives* 17(2):3–26.
- Hurd, Thomas R. 2016. *Contagion! Systemic risk in financial networks*. Springer.
- Kabanov, Yu M, R Mokbel and Kh El Bitar. 2018. “Clearing in Financial Networks.” *Theory of Probability & Its Applications* 62(2):252–277.
- Kemeny, John G and J Laurie Snell. 1976. *Markov chains*. Springer-Verlag, New York.
- Kusnetsov, Michael and Luitgard Anna Maria Veraart. 2019. “Interbank clearing in financial networks with multiple maturities.” *SIAM Journal on Financial Mathematics* 10(1):37–67.
- Liu, Ming and Jeremy Staum. 2010. “Sensitivity analysis of the Eisenberg–Noe model of contagion.” *Operations Research Letters* 38(5):489 – 491.
- Presman, Ernst. 2011. “A New Approach to the Solution of Optimal Stopping Problem in Discrete Time.” *Stochastics* 83(4–6):467–475.
- Puterman, Martin L. 2014. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons.
- Rogers, L. C. G. and L. A. M. Veraart. 2013. “Failure and Rescue in an Interbank Network.” *Management Science* 59(4):882–898.
- Roukny, Tarik, Stefano Battiston and Joseph E. Stiglitz. 2018. “Interconnectedness as a source of uncertainty in systemic risk.” *Journal of Financial Stability* 35:93 – 106.
- Shin, Hyun Song. 2008. “Risk and liquidity in a system context.” *Journal of Financial Intermediation* 17(3):315 – 329.
- Shin, Hyun Song. 2010. *Risk and Liquidity*. Number 9780199546367 in “OUP Catalogue” Oxford University Press.

- Shiryayev, Albert. 2019. *Probability-2*. Graduate Texts in Mathematics Springer, New York.
- Sonin, Isaac. 1999. “The state reduction and related algorithms and their applications to the study of Markov chains, graph theory, and the optimal stopping problem.” *Advances in Mathematics* 145(2):159–188.
- Sonin, Isaac M. 2006. The optimal stopping of a Markov chain and recursive solution of Poisson and Bellman equations. In *From Stochastic Calculus to Mathematical Finance*. Springer pp. 609–621.
- Sonin, Isaac M. 2008. “A generalized Gittins index for a Markov chain and its recursive calculation.” *Statistics & Probability Letters* 78(12):1526–1533.
- Suzuki, Teruyoshi. 2002. “Valuing corporate debt: the effect of cross-holdings of stock and debt.” *Journal of the Operations Research Society of Japan* 45(2):123–144.
- Veraart, Luitgard Anna Maria. 2020. “Distress and default contagion in financial networks.” *Mathematical Finance* 30(3):705–737.
- Yellen, Janet. 2013. “Interconnectedness and Systemic Risk: Lessons from the Financial Crisis and Policy Implications.” A speech at the American Economic Association/American Finance Association Joint Luncheon, San Diego, California, January 4, 2013.

# Appendix

## A1 Proof of Theorem 1(b)

In this subsection, we prove the monotonicity property of rates, which is equivalent to part (b) of Theorem 1. We prove by induction on  $k$ , where  $k$  is the index of the interval  $\Delta_k = [T_k, T_{k+1})$ ,  $k = 1, 2, \dots$ . We know that generally group  $J_0(t)$  changes in time: it can increase, decrease, become an empty set, and reappear again. When  $k = 1$ ,  $u_i = 1$  for all  $i$  and, as we explained in Subsection 4.2, at the moment of the first appearance of group  $J_0 = \{j\}$ , the value of the out-rate for member  $j$  of this group is  $v_j = u_j < 1$ . Suppose this statement is true on the status interval  $\Delta_k, k > 0$ . As before, we skip subindex  $k$ , denoting  $u_{i,k} = u_i, \Delta_k = \Delta, T_k = T, T_{k+1} = T'$ . We denote other vectors, coordinates, elements of partition, and matrices before the status change, i.e., on the interval  $\Delta_k$ , as  $J_0 = J_{0,k}$ , and the values after the change as  $J'_0 = J_{0,k+1}$ , etc. We denote the solution of the equilibrium equation on the interval  $\Delta_k$  as  $(v_i)$ , and on the interval  $\Delta_{k+1}$  as  $(w_i)$ . Thus, our induction statement is  $0 < v_i < 1, i \in J_0$ , and our goal is to prove that  $v_i \geq w_i$  for all  $i \in J'_0$ .

The status change at moment  $T_k = T$  occurs when  $b_r(T) = 0$  or  $c_r(T) = 0$ , where  $r \in J_+ \cup J_0$ . If  $b_r(T) = 0$  for  $r \in J_+$ , this means that the zero group is unchanged but the previous input vector  $(e_i)$  can decrease if bank  $r$  was a sponsor of a zero group. Then according to point (c) of Lemma 5.1, the new solution of the equilibrium equation can only strictly decrease. More difficult are cases when  $c_r(T) = 0$  for  $r \in J_+$  and when  $b_r(T) = 0$  for  $r \in J_0$ . In the first case,  $J'_0 = J_0 \cup r$ , and if  $|J_0| = m$ , the new system has  $m + 1$  variables, with value  $u_r = 1$  changed to a new unknown value  $w_r$ . In the second case, the new equilibrium system has only  $m - 1$  variables. We prove only the first case; the second case can be treated in a similar way. We consider a slightly more general statement for the following situation that includes our first case: suppose we have two partitions, an initial  $P = (J_+, J_0, J_*)$ , such that the positive banks have a subset  $L$  with  $n_i < 1, i \in L$ , and a new partition  $P' = (J'_+ = J_+ \setminus L, J'_0 = J_0 \cup L, J_*)$ .

**Lemma A1.1.** *Under the assumptions above, let  $v_i = n_i$  be the solution of the equilibrium system for partition  $P$ , and  $w_i = n'_i$  be the solution of the equilibrium system for partition*

$P'$ . Then  $v_i \geq w_i$  for all  $i \in J'_0$ , and  $n_i \geq n'_i$  for all  $i$ .

A heuristic proof is as follows. When the banks in  $L$  are classified as positive, their out-rates are equal to one, i.e., the maximum possible; when they are included as members of the zero group, the input vector into the zero group decrease, and since all obligations are the same, as a result, the in-rates (out-rates) in the new zero group become lower. Then all the new in-rates are lower. The technical difficulty in a rigorous proof is that we have to compare the solutions of two equilibrium systems of different size. It is sufficient to consider only the case when  $|L| = 1$ . Let  $L = \{r\}$ ,  $n_r < 1$ .

*Proof.* The equilibrium system (8) for  $w_i$ ,  $i \in J'_0 = J_0 \cup r$ , is

$$\begin{aligned} w_i &= e'_i + \sum_{j \in J_0} w_j q_{ji} + w_r q_{ri}, \quad i \in J_0, \\ w_r &= e'_r + \sum_{j \in J_0} w_j q_{jr}, \end{aligned} \tag{A1}$$

where  $e'_i = \sum_{j \in (J_+ \setminus r)} q_{ji} = \sum_{j \in J_+} q_{ji} - q_{ri} = e_i - q_{ri}$  and  $e'_r = \sum_{j \in J_+} q_{jr} = e_r$ .

The system for  $v_i$ ,  $i \in J_0$ , is  $v_i = e_i + \sum_{j \in J_0} v_j q_{ji}$ . We can rewrite this system to make it similar to (A1), using the artificial variable  $v_r = 1$  and the equalities  $e'_i = e_i - q_{ri}$  and  $e_r = e'_r$ , as

$$v_i = e'_i + \sum_{j \in J_0} v_j q_{ji} + v_r q_{ri}, \quad i \in J_0; \quad 1 = v_r = e'_r + a_r + \sum_{j \in J_0} v_j q_{jr}, \tag{A2}$$

where  $a_r = 1 - (e'_r + \sum_{j \in J_0} v_j q_{jr})$ . We know that this system has a solution  $v_i < 1$ ,  $i \in J_0$ ,  $v_r = 1$ . Now we want to compare the solutions of systems (A1) and (A2), which have the same size, to show that  $v_i \geq w_i$ ,  $i \in (J_0 \cup r)$  and  $w_r < 1$ . To use part (c) of Lemma 5.1, we need to show only that  $a_r \equiv 1 - (\sum_{j \in J_+} q_{jr} + \sum_{j \in J_0} v_j q_{jr}) > 0$ . We assumed that  $n_r = n_r(T_-) < 1$ . But  $n_r(T_-) = \sum_{j \in J_+} q_{jr} + \sum_{j \in J_0} v_j q_{jr} = 1 - a_r < 1$ , and therefore  $a_r > 0$ . Then, using part (c) of Lemma 5.1, we obtain that  $v_i \geq w_i$  for all  $i \in J_0$  and that  $1 > w_r$ , i.e.,  $w_i < 1$  for all  $i \in J'_0$ . Since the out-rates in the positive group remain the same, and the out-rates in the new zero group, including the new member  $r$ , are lower, then all in-rates are lower.

The second case, when  $b_r(T) = 0$ ,  $r \in J_0$ , and the new equilibrium system has only

smaller number of variables, can be proved similarly. Thus, we proved part (b) of Theorem 1, or, equivalently, the monotonicity property for in-rates and out-rates.  $\square$

## A2 The Big Bang Effect

The initial moment of time,  $t = 0$ , is exceptional if at this moment set  $J_0(0)$  is not empty. The reason for that is as follows. If at time  $t = 0$  all banks have a positive amount of cash, then  $u_i(t) = 1$  for all banks on some interval  $[0, T_1)$ , where moment  $T_1$  is the first moment when one of the banks either pays off its debt or its cash position hits zero level. After that, we can apply the monotonicity property, i.e., part (b) of Theorem 1. However, if at time zero there are some zero banks, then, informally, there is a Catch-22 situation. Some zero banks at moment  $t = 0$  should be classified as positive if the input rates for them exceed one. This means that at the “next moment after zero” they instantly become positive. Equivalently, for these banks, the balance rate  $d_i(0) > 0$ , or equivalently  $c'_i(t)_{t=0} > 0$ . But to find the input rates for all banks by solving equation (8), we need to know which banks are positive and which ones are in the zero group. In some cases, like in our Example 1, where only one bank had the initial cash position of zero and an in-rate greater than one, it was easy to reclassify this bank as positive. If there are two or more zero banks, then such reclassification cannot be done so easily. One way to circumvent this problem is as follows.

Assume that all zero banks received a small positive amount  $\varepsilon$  prior to the zero moment and then, during the small “probe” time interval  $\Delta_0 = [0, t_m)$ , having a length of order  $\varepsilon$ , we have a regular case analyzed in Subsection 4. Then during this interval, each of the initially zero banks will “reveal” its real status: the cash positions of the “real-zero” banks, having negative balance rates  $d_i(t)$ , very soon hit zero level during a series of close moments  $t_1, \dots, t_m$ , with  $m$  potentially equal zero. “Real-positive” banks will remain positive on the small interval  $(0, t_m)$  and will remain positive at least until a more distant moment  $T_1$ . This means that on a very short interval, we may have moments  $t_1, t_2, \dots, t_m$  of fast status changes, and at  $t_m$  all the real statuses are revealed.