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Strategic Approximations of Discontinuous Games

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Abstract

An infinite game is approximated by restricting the players to finite subsets of their pure strategy spaces. A *strategic approximation* of an infinite game is a countable subset of pure strategies with the property that limits of all equilibria of all sequences of approximating games whose finite strategy sets eventually include each member of the countable set must be equilibria of the infinite game. We provide conditions under which infinite games admit strategic approximations.

Keywords: discontinuous games, finite approximation
JEL Classification: C7

1. Introduction

The analytic convenience of infinite strategy spaces has often proven to be of value in the analysis of games. But when the presence of infinitely many strategies is crucial – such as when discontinuities play a central role and cannot be eliminated or smoothed – doubts may arise over the robustness or even the relevance of the results. One way to attenuate such doubts is to provide a sequence of finite approximating games whose equilibria converge to equilibria of the infinite game, where each approximating game restricts the players to finite subsets of their strategy spaces. This approach becomes even more convincing when there is some robustness with respect to the choice of approximating games.

As observed by Simon (1987), a game with infinite strategy spaces might include strategies that are of particular strategic significance to the players.¹ When this is the case, one cannot hope to well-approximate the infinite game without eventually including such strategies in the approximating games. Thus, on the one hand, good approximations cannot be arbitrary –

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¹Consider, for example, the strategy to produce zero for a firm with fixed costs that must be paid only when production is positive.

they must include strategically important strategies. On the other hand, once such strategies are identified for eventual inclusion, the approximating games – i.e., the sequence of finite strategy sets – ought otherwise be arbitrary so as to establish the irrelevance of the remaining details of the approximating sequences.

A *strategic approximation* is a countable subset of pure strategies with the property that limits of all equilibria of all sequences of approximating games whose finite strategy sets eventually include each member of the countable set must be equilibria of the infinite game. Our objective is to provide conditions under which strategic approximations exist.²

2. Preliminaries

We maintain the following assumptions throughout the paper. There are N players. Player i has pure strategy set X_i , a nonempty compact metric space. We let $X = \times_i X_i$ and endow all product sets with the product topology. Player i 's von Neumann-Morgenstern utility function, $u_i : X \rightarrow \mathbb{R}$, is bounded and measurable. This defines a game $G = (X_i, u_i)_{i=1}^N$.

Let M_i denote the space of probability measures on the Borel subsets of X_i and let $M = \times_i M_i$. Extend each u_i to M by $u_i(m_1, \dots, m_N) = \int_X u_i(x) dm_1 \dots dm_N$. The *mixed extension* of G is $\bar{G} = (M_i, u_i)_{i=1}^N$. The space M is a compact metric space when endowed with the Prohorov metric (see Billingsley (1968)). By *equilibrium* we shall always mean a mixed strategy Nash equilibrium of the game under consideration.

$G' = (X'_i, v_i)_{i=1}^N$ is an *approximation* of $G = (X_i, u_i)_{i=1}^N$ if each X'_i is a finite subset of X_i and v_i is the restriction of u_i to $\times_i X'_i$. From now on, we will write u_i instead of v_i , so that $G' = (X'_i, u_i)_{i=1}^N$. Thus, an approximating game simply restricts the players to finite subsets of their pure strategy sets. The following concept will figure prominently here.

Definition 2.1. (Reny (2009)). *The game G has the **finite deviation property** if whenever $m \in M$ is not an equilibrium of G , there is a neighborhood U of m and a finite subset D of M such that for every $m' \in U$ there is a player i and $\hat{m} \in D$ such that $u_i(\hat{m}_i, m'_{-i}) > u_i(m')$. The set D is called a **finite deviation set** for U . If in addition the members of D can always be chosen to have finite supports we say that G has the **finite-support finite deviation property**.*

Reny (2009) observes that if G has the finite deviation property then G possesses a mixed strategy equilibrium and demonstrates that G has the finite deviation property if its mixed extension is better-reply secure. Because of the connection to better-reply security, we remind the reader of the definition.

²A rather distinct approach to approximating infinite games can be pursued by exploiting the techniques introduced in Simon and Zame (1990).

A pair $(m, u) \in M \times \mathbb{R}^N$ is in $\bar{\Gamma}$, the closure of the graph of the vector payoff function of $\bar{G}$, if $u = \lim_n (u_1(m^n), \dots, u_N(m^n))$ for some sequence m^n of mixed strategies in M converging to m . Following Reny (1999), say that player i can *secure* the payoff α at $m \in M$ in the mixed extension of G , if player i has a strategy $\hat{m}_i \in M_i$ such that $u_i(\hat{m}_i, m'_{-i}) \geq \alpha$ for all m'_{-i} in some neighborhood of m_{-i} . We then say that $\hat{m}_i$ *secures the payoff* α at m for i .

Definition 2.2. (Reny (1999)). *The mixed extension of G is **better-reply secure** if whenever (m, u) is in $\bar{\Gamma}$ and m is not an equilibrium, some player i can secure a payoff strictly greater than u_i at m in the mixed extension of G . If in addition the securing (mixed) strategies can always be chosen to have finite supports, then the mixed extension of G is **finite-support better-reply secure**.*

For later reference, we state the following result.

Theorem 2.3. (Reny (2009)). *If the mixed extension of G is better-reply secure then G has the finite deviation property. Moreover, if the mixed extension of G is finite-support better-reply secure then G has the finite-support finite deviation property.*³

3. Strategic Approximations

Our central definition is the following.⁴

Definition 3.1. A **strategic approximation** of a game $G = (X_i, u_i)_{i=1}^N$ is a countable set of pure strategies $X^\infty = \times_{i=1}^N X_i^\infty$ contained in X , such that whenever for each player i , $X_i^1 \subseteq X_i^2 \subseteq \dots$ is an increasing sequence of finite subsets of X_i whose union contains X_i^∞ , any limit of equilibria of the sequence of finite games $(X_i^1, u_i)_{i=1}^N, (X_i^2, u_i)_{i=1}^N, \dots$ is an equilibrium of G .

Several examples illustrate the main ideas.⁵

3.1. Existence of a Strategic Approximation

There is one player whose payoff is 1 if he chooses $x = 0$ and is zero if he chooses any other $x \in [0, 1]$. The unique equilibrium is $x = 0$. However, any sequence of finite approximations

³Reny (2009) does not prove the latter result but it follows immediately from Reny's proof of the former.

⁴We include finite sets among those which are considered countable (see also Royden (1968)). Thus, X^∞ below may be finite.

⁵Note that because the set of mixed strategies, M , is sequentially compact, the existence of a strategic approximation of G implies that G possesses an equilibrium in mixed strategies. Each finite game in the approximating sequence possesses a mixed strategy equilibrium by Nash's (1950) theorem, and the sequential compactness of M ensures that some such sequence converges.

whose strategy sets become dense in $[0, 1]$ and which do not eventually contain the strategy $x = 0$ admits sequences of equilibria that converge to any element of $[0, 1]$. Consequently, “good” approximations must include the strategy $x = 0$. Moreover, if $X^1, X^2, \dots$ is any sequence of finite subsets of $[0, 1]$ such that $0 \in X^n$ for n large enough, then the limit of any sequence of equilibria of the finite games converges to $x = 0$. This game therefore possesses a strategic approximation, namely $X^\infty = \{0\}$.⁶

3.2. Nonexistence of a Strategic Approximation I.

The following two-person example is taken from Simon (1987), and we encourage the reader to consult that paper for the details of arguments omitted here. Player 1 chooses $x \in [0, 1]$ and player 2 chooses $y \in [0, 1]$. A player’s payoff is 1 if his choice is positive and equal to one-half of his opponent’s choice or if his choice is 1 and his opponent’s choice is zero. Otherwise, a player’s payoff is zero.

It is not difficult to see that this game possesses no pure strategy equilibrium. It is not much more difficult to see that there are no mixed equilibria in which either player assigns positive probability to any pure strategy. Thus, the only possible equilibria are those in which each player employs an atomless mixed strategy. Finally, any pair of atomless mixed strategies is easily seen to be an equilibrium because given the opponent’s strategy a player receives an expected payoff of zero regardless of the pure strategy he employs.

If one attempts to approximate this infinite game, then regardless of the finite subsets of pure strategies one employs, the approximation will possess a pure strategy equilibrium. For example, suppose that both players’ finite pure strategy sets contain positive elements of $[0, 1]$ and player 1’s smallest positive strategy is x' and 2’s is y' where $x' \leq y'$. Then, either (x', y') is an equilibrium or player 1’s finite strategy set contains the strategy $y'/2$ in which case $(y'/2, y')$ is an equilibrium. But if every finite approximation contains a pure strategy equilibrium then any limit of such equilibria will be pure and so will not be an equilibrium of the infinite game. This game therefore fails to possess a strategic approximation.

3.3. Nonexistence of a Strategic Approximation II.

Consider the following two-person zero-sum game. Player 1 chooses $x \in [0, 1]$ and player 2 chooses $y \in [0, 1]^\infty$. Player 1’s payoff is one if his choice does not match any of the coordinates of 2’s choice and is zero otherwise.⁷ Clearly, player 1 can guarantee a payoff of 1 if and only

⁶In this very special example, equilibria of the approximating games that include $x = 0$ are themselves equilibria of the original game. One cannot hope to obtain this property in general.

⁷Assigning $[0, 1]$ its usual metric and employing the product topology on $[0, 1]^\infty$, the strategy spaces are compact and the payoff function is Borel measurable.

if he employs an atomless mixed strategy. Consequently, a strategy pair is an equilibrium if and only if player 1's strategy is mixed and atomless.

Consider the following natural attempt at approximating this game. For $n = 1, 2, \dots$, G_n denotes the n -th approximating game in which player 1's finite pure strategy set is,

$$X_n = \left\{ \frac{1}{n^2}, \frac{2}{n^2}, \dots, \frac{n^2}{n^2} \right\},$$

and player 2's finite pure strategy set is,

$$Y_n = (X_n)^n \times \{(1, 1, 1, \dots)\}.$$

Thus in the n -th approximating game player 1 has n^2 strategies and player 2 can choose any of the finitely many vectors in $[0, 1]^\infty$ whose first n coordinates are each a member of 1's strategy set X_n and whose remaining coordinates are equal to 1. It is readily verified that all equilibria of G_n are in mixed strategies of the following form. Player 1 mixes uniformly among the members of X_n and player 2 mixes among the members of Y_n so that (i) each member of X_n is equally likely to appear as one of the first n coordinates of the pure strategy realization, and (ii) the first n coordinates of any member of Y_n assigned positive probability are distinct.⁸ Taking the limit as $n \rightarrow \infty$ yields, upon extraction of a weak*-convergent subsequence if necessary, limit strategies in which player 1 employs Lebesgue measure on $[0, 1]$ and player 2 employs some mixed strategy – i.e., a probability measure on the Borel subsets of $[0, 1]^\infty$.⁹ Because Lebesgue measure is atomless, this limit is an equilibrium of the original infinite game.

Even though any limit of equilibria of the particular sequence of approximating games G_n is an equilibrium of the original infinite game, the original game fails to possess a strategic approximation because approximations of it are not robust to the inclusion of additional strategies. For example, providing player 2 with additional strategies in G_n by instead defining $Y_n = (X_n)^{n^2} \times \{(1, 1, 1, \dots)\}$ would permit player 2 to achieve his highest possible payoff of zero in every equilibrium of G_n by choosing, for example, the pure strategy $\frac{1}{n^2}(1, 2, \dots, n^2, 1, 1, 1, \dots)$, against which player 1 cannot avoid matching one of the coordinates. This strategy for player 2 paired with any strategy for player 1 constitutes an equilibrium of G_n . Moreover, if in each G_n player 1's strategy is, for example, pure then no limit of such strategies is an equilibrium of the original game. Every attempt to approximate this infinite game will have a similar defect. Hence, this game fails to possess a strategic ap-

⁸For example, for each $k = 0, 1, \dots, n-1$ player 2 can assign probability $1/n$ to the vector $\frac{1}{n^2}(kn+1, kn+2, \dots, kn+n, 0, 0, \dots)$.

⁹Because 2's strategy space is compact in the product topology a weak*-convergent subsequence is guaranteed to exist.

proximation because adding additional strategies to any approximation of it can produce approximating-game equilibria that are far from any of its equilibria.

4. Strategic Approximations of Mixed Extensions

As a preliminary step, we apply the definition of a strategic approximation to the mixed extension, $\bar{G} = (M_i, u_i)_{i=1}^N$, of G . Evidently, a strategic approximation of $\bar{G}$ is a countable subset $M^\infty = \times_{i=1}^N M_i^\infty$ of M such that whenever, for each player i , $M_i^1 \subseteq M_i^2 \subseteq \dots$ is an increasing sequence of finite subsets of M_i whose union is M_i^∞ , any limit of (mixed strategy) equilibria of the sequence of approximating games $(M_i^1, u_i)_{i=1}^N, (M_i^2, u_i)_{i=1}^N, \dots$, is a (mixed strategy) equilibrium of $\bar{G}$ (and hence its distribution on X is a mixed strategy equilibrium of G).

A first indication of the relevance of the finite deviation property to the existence of strategic approximations is the following.

Theorem 4.1. *If G has the finite deviation property, then the mixed extension of G has a strategic approximation.*

Proof. If G has the finite deviation property, then every non equilibrium point of G is contained in an open set of non equilibrium points. Hence, the set of non equilibrium points of G , U say, is open. We note that, being an open subset of a compact metric space, every open cover of U has a countable subcover.¹⁰

By the finite deviation property, for every $m \in U$ there is a neighborhood U^m of m and a finite subset D^m of M such that D^m is a finite deviation set for U^m . This yields a collection of pairs (U^m, D^m) , one pair for each $m \in U$.

Because the U^m form an open cover of U , there is a countable subcollection, $\{(U^1, D^1), (U^2, D^2), \dots\}$ of $\{(U^m, D^m)\}_{m \in U}$ such that $U = \cup_{k=1}^\infty U^k$. For each player i and every $k = 1, 2, \dots$, let D_i^k be the projection of D^k onto M_i and define $M_i^\infty = \cup_{k=1}^\infty D_i^k$. It suffices to show that the countable set $M^\infty = \times_{i=1}^N M_i^\infty$ is a strategic approximation of $\bar{G}$.

For each player i , let $M_i^1 \subseteq M_i^2 \subseteq \dots$ be an increasing sequence of finite subsets of M_i whose union contains M_i^∞ , and for each n let μ^n be a mixed strategy equilibrium of the finite game $\bar{G}^n = (M_i^n, u_i)_{i=1}^N$. Thus, for example, if $m_i \in M_i^n$, then $\mu_i^n(m_i)$ is the probability that player i assigns to m_i in the equilibrium μ^n of $\bar{G}^n$.¹¹ Suppose that for each player i , $\mu_i^n \rightarrow \mu_i^* \in \Delta(M_i)$, where $\Delta(M_i)$ denotes the set of probability measures on the Borel subsets

¹⁰To see this, note that U is the union over $n = 1, 2, \dots$ of the countably many compact sets $C_n = \{m \in U : B_n(m) \subseteq U\}$, where $B_n(m)$ is the open ball of radius $1/n$ around m . For any open cover of U there is for each n a finite subcover of C_n and their union is a countable subcover of U .

¹¹The m_i 's are thus treated as pure strategies in the finite games that approximate the mixed extension of G even though they are mixed strategies in the original game G .

of M_i . (Because M_i is compact metric, $\Delta(M_i)$ is also compact metric; see e.g., Billingsley (1968)). We must show that μ^* is an equilibrium of $\bar{G}$.

For each player i and every n , let $m_i^n = \sum_{m_i \in M_i^n} \mu_i^n(m_i) m_i$, and suppose, without loss by sequential compactness, that $m_i^n \rightarrow m_i^* \in M_i$. By definition, m^n is the distribution of μ^n on X . That is, for every n and every Borel subset B of X , $m^n(B) = \int_M m(B) d\mu^n(m)$ – note that the integral is in fact a sum because μ^n has finite support. By Lemma 7.1 in the appendix, the limit m^* of m^n is the distribution on X of the limit μ^* of μ^n .

We claim that m^* is an equilibrium of G . If not, then $m^* \in U$ and there exists (U^k, D^k) such that $m^* \in U^k$ and $D^k \subseteq M$ is a finite deviation set for U^k . Because $m^n \rightarrow m^*$, we have $m^n \in U^k$ for n large enough, so that for all such n some player i_n can profitably deviate from m^n by employing a strategy $\hat{m}_{i_n}^n$ in $D_{i_n}^k$. Consequently, for all n large enough,

$$\begin{aligned} u_{i_n}(\hat{m}_{i_n}^n, \mu_{-i}^n) &= u_{i_n}(\hat{m}_{i_n}^n, m_{-i}^n) \\ &> u_{i_n}(m^n) \\ &= u_{i_n}(\mu^n),^{12} \end{aligned}$$

where the first and third lines follow because mixtures over mixtures over X are payoff equivalent to their distributions over X .

For n sufficiently large, $D_{i_n}^k$ is contained in $M_{i_n}^n$ so that $\hat{m}_{i_n}^n$ is a feasible strategy for player i_n in $\bar{G}_n$. But then $u_{i_n}(\hat{m}_{i_n}^n, \mu_{-i}^n) > u_{i_n}(\mu^n)$ for all n large enough contradicts the fact that μ^n is an equilibrium of $\bar{G}^n$. We conclude that m^* is an equilibrium of G . But then μ^* is an equilibrium of $\bar{G}$ because for every player i and every $m_i \in M_i$,

$$\begin{aligned} u_i(m_i, \mu_{-i}^*) &= u_i(m_i, m_{-i}^*) \\ &\leq u_i(m^*) \\ &= u_i(\mu^*), \end{aligned}$$

where, once again, the first and third lines hold because mixtures over mixtures are payoff equivalent to their distributions over X , and the second line follows because m^* is an equilibrium of G . ■

Unfortunately G need not possess a strategic approximation even though its mixed extension does, as the following example demonstrates.¹³

¹²For $\mu \in \Delta(M)$, $u_i(\mu) = \int_M u_i(m) d\mu$.

¹³The infinite game in example II does not furnish such an example because its mixed extension does not possess a strategic approximation. To see this, note that for any approximation of its mixed extension one can add a pure strategy to player 1's strategy set that, according to each of the finitely many mixed strategies available to player 2, occurs with probability zero in each coordinate of 2's realized pure strategy.

4.1. Nonexistence of a Strategic Approximation III.

There are three players. Players 1 and 2 participate in a first-price all-pay auction with a uniform tie-break rule in which it is common knowledge that the object at auction is worth one dollar to each of them. Each player can submit a bid from $[0,1]$ – player 1 chooses x and player 2 chooses y – and they are each risk neutral.

Player 3 chooses $z \in [0,1]^\infty$ and always receives a payoff of zero regardless of the choices of x , y and z . If any coordinate of z is equal to 1's choice of x , then the outcome of the auction is null and void – i.e., neither player 1 nor player 2 pays his bid and neither player wins the object – and in addition player 1 loses a dollar. Otherwise, player 3's choice has no effect.

Ignoring for the moment the presence of player 3, a standard argument establishes that there is a unique equilibrium in the all-pay auction, namely that both players 1 and 2 independently randomize according to Lebesgue measure on $[0,1]$. Given these strategies, player 3's presence has no effect on expected payoffs because regardless of 3's choice of z , the probability that one of its coordinates matches 1's choice of x is zero.

In fact, every equilibrium of the 3-player game is such that players 1 and 2 choose according to Lebesgue measure, while player 3's choice of mixed strategy can be arbitrary. This characterization follows from the fact that for any strategy of player 2, player 1 can achieve a payoff arbitrarily close to a best reply in the auction by employing an atomless strategy. Consequently, player 1 can, with probability one, ensure that his choice not match any coordinate of 3's choice while simultaneously achieving a payoff in the auction that is arbitrarily close to maximal given 2's strategy. Therefore, in any equilibrium of the 3-player game the results of the auction will stand with probability one and player 1's strategy must be a best reply in the auction. Since the results of the auction stand with probability one, player 2's strategy must also be a best reply in the auction and the argument is complete.

For the same reason as in the previous example, this game fails to possess a strategic approximation. Regardless of the attempted strategic approximation, adding to player 3's pure strategy set a pure strategy whose coordinates contain every pure strategy available to player 1 creates an equilibrium in which player 3 chooses that strategy and players 1 and 2 each choose any pure strategy. Limits of such strategies, being pure for both players 1 and 2, are not equilibria of the original game.

Finally, it can be shown that this game's mixed extension is better-reply secure and so by Theorem 2.3 this game has the finite deviation property. Consequently, the mixed extension of this game (by Theorem 4.1), but not the game itself, possesses a strategic approximation.¹⁴

This creates a pure strategy equilibrium of the approximating game in which player 1 employs the added pure strategy. The limit of such equilibria, being pure for player 1, is not an equilibrium of the infinite game.

¹⁴The example neither relies on three players nor on infinite-dimensional strategy spaces. All of the

Our next result shows that the discrepancy is due to the failure of the game to possess the finite-support finite deviation property.

5. Strategic Approximations of G

Our first result on the existence of strategic approximations of G follows the same line of reasoning as in the proof of Theorem 4.1.

Theorem 5.1. *If G has the finite-support finite deviation property, then G has a strategic approximation.*

Proof. Maintain the same notation (i.e., U , U^m , D^m etc.) and construct M^∞ as in the proof of Theorem 4.1. By the finite-support finite deviation property, we may assume that each member of M^∞ has finite support. For each player i , let X_i^∞ be the countable set that is the union of the supports of all the members of M_i^∞ and let $X^\infty = \times_{i=1}^N X_i^\infty$. We wish to show that X^∞ is a strategic approximation of G .

For each player i , let $X_i^1 \subseteq X_i^2 \subseteq \dots$ be an increasing sequence of finite subsets of X_i whose union contains X_i^∞ , and for each n let m^n be a mixed strategy equilibrium of the finite game $G^n = (X_i^n, u_i)_{i=1}^N$ and suppose that $m^n \rightarrow m^* \in M$. We must show that m^* is an equilibrium of G .

If not, then $m^* \in U$ and there exists (U^k, D^k) such that $m^* \in U^k$ and $D^k \subseteq M$ is a finite deviation set for U^k . Because $m^n \rightarrow m^*$, we have $m^n \in U^k$ for n large enough, so that for all such n some player i_n can profitably deviate from m^n by employing a strategy $\hat{m}_{i_n}^n$ in $D_{i_n}^k$. Moreover, because the union of the supports of the members of $D_{i_n}^k$ is finite this union is contained in $X_{i_n}^n$ for n large enough, implying that the profitable deviation $\hat{m}_{i_n}^n$ is feasible for player i_n in $G^n = (X_i^n, u_i)_{i=1}^N$. This contradicts the fact that m^n is an equilibrium of G^n and we conclude that m^* is an equilibrium of G . ■

An immediate consequence of Theorems 2.3 and 5.1 is the following.

Corollary 5.2. *If the mixed extension of G is finite-support better-reply secure, then G has a strategic approximation.*

Both Dasgupta and Maskin (1986) and Simon (1987) use approximation techniques to establish the existence of an equilibrium. As shown in Reny (1999), their hypotheses imply

features of this example can be obtained with 2 players and pure strategy sets $[0,1]$ and $[0,1]^2$. The idea is to give player 2 a second coordinate dummy choice and to map his second coordinate $[0,1]$ onto $\cup_k [0,1]^k$ using the Peano curve construction. This again results in measurable payoffs and the dummy choice can be used against player 1 as in the three-player game.

that the game's mixed extension is better-reply secure. Reny's discussion in fact demonstrates that their hypotheses imply finite-support better-reply security. Consequently, the hypotheses of Corollary 5.2 weaken those of both Dasgupta and Maskin (1986) and Simon (1987), and so their hypotheses too imply the existence of a strategic approximation.¹⁵

Our final two results provide sufficient conditions for a better-reply secure game to be finite-support better-reply secure and hence to have a strategic approximation. The second of the two results is implied by the first, but its hypotheses have the advantage of being stated in terms of the primitives of the game. The hypotheses of the two results are satisfied in many economic games (Bertrand, Hotelling, auctions, etc.).

Theorem 5.3. *Suppose that the mixed extension of $G = (X_i, u_i)_{i=1}^N$ is better-reply secure. If for every $m \in M$ and every player i , each payoff below a payoff that i can secure at m can be secured there by a mixed strategy for i whose atomless part, for all x_{-i} , assigns probability zero to the set of x_i such that $u_i(\cdot)$ is discontinuous at (x_i, x_{-i}) , then G admits a strategic approximation.*

Proof. For each player i and every $m \in M$, define $\underline{u}_i(m_i, m_{-i}) = \liminf_{m'_{-i} \rightarrow m_{-i}} u_i(m_i, m'_{-i})$. Consequently, player i can secure the payoff α_i at $m \in M$ if and only if $\underline{u}_i(\hat{m}_i, m_{-i}) \geq \alpha_i$ for some $\hat{m}_i \in M_i$.

Fix a player i , $m \in M$, and $\varepsilon > 0$. Clearly, the strategy m_i secures the payoff $\underline{u}_i(m_i, m_{-i})$ at m . Hence by hypothesis there is a strategy $\hat{m}_i \in M_i$ securing the payoff $\underline{u}_i(m_i, m_{-i}) - \varepsilon$ at m – i.e., $\underline{u}_i(\hat{m}_i, m_{-i}) \geq \underline{u}_i(m_i, m_{-i}) - \varepsilon$ – and such that the atomless part of $\hat{m}_i$, for all x_{-i} , assigns probability zero to the set of x_i such that $u_i(\cdot)$ is discontinuous at (x_i, x_{-i}) .

Decompose $\hat{m}_i$ into $\hat{m}_i = \bar{\nu}_i + \bar{\lambda}_i$, where $\bar{\nu}_i$ is the atomless part of $\hat{m}_i$, and $\bar{\lambda}_i$ is concentrated on a countable set. Assume that both $\bar{\nu}_i(X_i)$ and $\bar{\lambda}_i(X_i)$ are nonzero.¹⁶ Define the probability measures $\nu_i = \bar{\nu}_i / \bar{\nu}_i(X_i)$ and $\lambda_i = \bar{\lambda}_i / \bar{\lambda}_i(X_i)$ on X_i . Since the probability measures on X_i with finite support are dense in M_i (see Billingsley (1968)), there is a sequence $\{\nu_i^k\}_{k=1}^\infty$ of elements of M_i having finite support such that $\nu_i^k \rightarrow \nu_i$.

For each $x \in X$, let $w_i(x) = \liminf_{x^n \rightarrow x} u_i(x^n)$. Consequently, w_i is real-valued (since $u_i(\cdot)$ is bounded) and lower semicontinuous on X . In addition, because $w_i(x) = u_i(x)$ whenever $u_i(\cdot)$ is continuous at x , we have by hypothesis that,

$$\nu_i\{x_i : w_i(x_i, x_{-i}) \neq u_i(x_i, x_{-i})\} = 0, \text{ for all } x_{-i} \in X_{-i},$$

¹⁵In fact, the hypotheses of Dasgupta and Maskin (1986) are sufficiently strong to guarantee that for any game satisfying them, any dense (under the Euclidean metric) subset of the players' Euclidean strategy spaces is a strategic approximation. For a related result see Section 6 below.

¹⁶The cases in which one of $\bar{\nu}_i$ or $\bar{\lambda}_i$ is zero are handled by applying the same argument as follows but without the normalization.

so that

$$\int_{X_i} w_i(x_i, x_{-i}) d\nu_i = u_i(\nu_i, x_{-i}), \text{ for all } x_{-i} \in X_{-i}, \quad (5.1)$$

and the function of x_{-i} defined by both sides of (5.1) is continuous on X_{-i} .¹⁷

Because (ν_i^k, m_{-i}) converges weakly to (ν_i, m_{-i}) , the lower semicontinuity of $w_i(\cdot)$ implies,

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_X w_i(x_i, x_{-i}) d\nu_i^k dm_{-i} &\geq \int_X w_i(x_i, x_{-i}) d\nu_i dm_{-i} \\ &= u_i(\nu_i, m_{-i}), \end{aligned} \quad (5.2)$$

where the second line follows from (5.1). Hence for all k large enough,

$$\begin{aligned} \underline{u}_i(\nu_i^k, m_{-i}) &= \liminf_{m'_{-i} \rightarrow m_{-i}} u_i(\nu_i^k, m'_{-i}) \\ &\geq \liminf_{m'_{-i} \rightarrow m_{-i}} \int_X w_i(x_i, x_{-i}) d\nu_i^k dm'_{-i} \\ &\geq \int_X w_i(x_i, x_{-i}) d\nu_i^k dm_{-i} \\ &\geq u_i(\nu_i, m_{-i}) - \varepsilon, \end{aligned} \quad (5.3)$$

where the second line holds because $w_i(\cdot) \leq u_i(\cdot)$, the third line holds because $w_i(\cdot)$ is lower semicontinuous, and the fourth holds by (5.2).

Choose a sequence of finite-support strategies $\lambda_i^k \in M_i$, such that $\lambda_i^k(x_i) \rightarrow \lambda_i(x_i)$ for each of the countably many x_i given positive weight by λ_i . Because $u_i(\cdot)$ is bounded,

$$\underline{u}_i(\lambda_i^k, m_{-i}) \geq \underline{u}_i(\lambda_i, m_{-i}) - \varepsilon, \text{ for } k \text{ large enough.} \quad (5.4)$$

Letting $m_i^k = a\nu_i^k + (1-a)\lambda_i^k$, where $a = \bar{\nu}_i(X_i)$ (so that $1-a = \bar{\lambda}_i(X_i)$) gives, for k large enough,

$$\begin{aligned} \underline{u}_i(m_i^k, m_{-i}) &= \underline{u}_i(a\nu_i^k + (1-a)\lambda_i^k, m_{-i}) \\ &\geq a\underline{u}_i(\nu_i^k, m_{-i}) + (1-a)\underline{u}_i(\lambda_i^k, m_{-i}) \\ &\geq a(u_i(\nu_i, m_{-i}) - \varepsilon) + (1-a)(\underline{u}_i(\lambda_i, m_{-i}) - \varepsilon) \\ &= \liminf_{m'_{-i} \rightarrow m_{-i}} [a u_i(\nu_i, m'_{-i}) + (1-a) u_i(\lambda_i, m'_{-i})] - \varepsilon \\ &= \underline{u}_i(\hat{m}_i, m_{-i}) - \varepsilon, \\ &\geq \underline{u}_i(m_i, m_{-i}) - 2\varepsilon, \end{aligned}$$

¹⁷To see conitnuity, note that if $x'_{-i} \rightarrow x_{-i}$, then the sequence of functions $u_i(\cdot, x'_{-i})$ on X_i converges to the function $u_i(\cdot, x_{-i})$ pointwise for ν_i a.e. $x_i \in X_i$. Since $u_i(\cdot)$ is bounded, Lebesgue's dominated convergence theorem implies $u_i(\nu_i, x'_{-i}) \rightarrow u_i(\nu_i, x_{-i})$.

where the third line follows from (5.3) and (5.4), and the fourth line follows from the continuity of $u_i(\nu_i, m'_{-i})$ in m'_{-i} established in (5.1).

We have therefore demonstrated the following. For every player i , every $\varepsilon > 0$, and every $m \in M$, there exists $\bar{m}_i \in M_i$ with finite support such that

$$\underline{u}_i(\bar{m}_i, m_{-i}) \geq \underline{u}_i(m_i, m_{-i}) - 2\varepsilon,$$

from which it is immediate that, being better-reply secure, G is finite-support better-reply secure. The result now follows from Theorems 2.3 and 5.1. ■

An immediate consequence of Theorem 5.3 is the following.

Corollary 5.4. *Suppose that the mixed extension of $G = (X_i, u_i)_{i=1}^N$ is better-reply secure. If for each player i and every $x_{-i} \in X_{-i}$, $u_i(\cdot)$ is continuous at (x_i, x_{-i}) for all but perhaps countably many $x_i \in X_i$, then G admits a strategic approximation.*

Remark 1. *The countable discontinuity condition in Corollary 5.4 is strictly weaker than Dasgupta and Maskin's (1986) "diagonal discontinuity" restrictions on the set of discontinuities of the players' payoff functions.*

6. A Topological Perspective

Another view to approximating infinite games is to insist that every dense subset of the players' strategy sets constitutes a strategic approximation. This point of view therefore places emphasis on the underlying topology. A natural question then is, under what conditions does there exist a topology on the game's strategy space such that every countable dense subset of pure strategies is a strategic approximation? Note that examples II and III illustrate that such a topology need not exist. Nevertheless, one answer is as follows.

Theorem 6.1. *If $G = (X_i, u_i)_{i=1}^N$ has the finite-support finite deviation property, then there is a topology on each X_i such that whenever, for each i , X_i^∞ is a countable dense subset of X_i , $\times_i X_i^\infty$ is a strategic approximation of G .*

Proof. For each i , define X_i^∞ as in the proof of Theorem 5.1. Consequently, $\times_i X_i^\infty$ is a strategic approximation of G in the given topology. Let d_i be the given metric on X_i and define a new metric, δ_i , on X_i so that $\delta_i(x_i, x'_i)$ is equal to $d_i(x_i, x'_i)$ unless either x_i or x'_i is a member of X_i^∞ , in which case $\delta_i(x_i, x'_i) = 1$. The result now follows because convergence under δ_i implies convergence under d_i and under the metric δ_i every dense subset of X_i contains X_i^∞ .¹⁸ ■

¹⁸We remark that under the new metric, the space of pure strategies need no longer be compact.

7. Appendix

Lemma 7.1. Suppose $m^n \rightarrow m^*$ and $\mu^n \rightarrow \mu^*$, where m^* and each m^n are in $M = \Delta(X)$ and μ and each μ^n are in $\Delta(M)$. If

$$m^n(B) = \int_M m(B) d\mu^n(m),$$

holds for every Borel subset B of X and every n , then

$$m^*(B) = \int_M m(B) d\mu^*(m), \tag{7.1}$$

holds for every Borel subset B of X .¹⁹

Proof. Let $f : X \rightarrow [0, 1]$ be continuous. Then by the definition of m^n ,

$$\int_X f(x) dm^n(x) = \int_M \int_X f(x) dm(x) d\mu^n(m).$$

The left-hand side of this equality converges to $\int_X f(x) dm^*(x)$ and the right-hand side converges to $\int_M \int_X f(x) dm(x) d\mu^*(m)$ because the function $\int_X f(x) dm(x)$ is continuous in m . Hence, we may conclude that for every continuous $f : X \rightarrow [0, 1]$,

$$\int_X f(x) dm^*(x) = \int_M \int_X f(x) dm(x) d\mu^*(m).$$

A standard argument then establishes (7.1).²⁰ ■

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¹⁹Note that for each Borel subset B of X , the real-valued function of m defined by $m(B)$ is upper semi-continuous in m on M and hence measurable.

²⁰The standard argument considers sequences of continuous functions converging pointwise to indicator functions on open sets thereby establishing the desired equality by the dominated convergence theorem when B is open. But then the two measures on X defined by the left-hand and right-hand sides of (7.1) coincide on open sets and so must coincide on all Borel sets.

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