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ABSTRACT

We study equilibria in a heterogeneous-agent incomplete-market economy with nominal government debt and flexible prices. Unlike in representative agent economies, steady-state equilibria exist when the government runs persistent deficits, provided that the level of deficits is not too large. In these equilibria, the real interest rate is below the growth rate of the economy. We quantify the maximum sustainable deficit for the US and show that it is lower under more redistributive tax and transfer systems. With constant primary deficits, there exist two steady-states, and the price level and inflation are not uniquely determined. We describe alternative policy settings that deliver uniqueness. We conduct quantitative experiments to illustrate how redistribution and precautionary saving amplify price level increases in response to fiscal helicopter drops, deficit expansions, and loose monetary policy. We show that rising primary deficits can account for a decline in the long-run real interest rate, leading to higher inflation for any given monetary policy. Our work highlights the role of household heterogeneity and market incompleteness in determining inflation.

Greg Kaplan
Department of Economics
University of Chicago
1126 E 59th St
Chicago, IL 60637
and NBER
gkaplan@uchicago.edu

Giovanni L. Violante
Department of Economics
Princeton University
Julis Romo Rabinowitz Building
Princeton, NJ 08540
and NBER
violante@princeton.edu

Georgios Nikolakoudis
Princeton University
gn3@princeton.edu

1 Introduction

We develop a framework to study the causes and consequences of price level dynamics in an economy with three features:

- (i) a fiscal authority issues nominal debt to finance real expenditures and transfers to households;
- (ii) a monetary authority sets the short-term nominal interest rate on government debt;
- (iii) incomplete financial markets so that households have a precautionary motive to accumulate savings in order to self-insure against idiosyncratic income risk.

Our interest in economies with the first two features is motivated by institutional arrangements in the real world. Such economies have been extensively studied, most recently under the label *Fiscal Theory of the Price Level* (FTPL).¹ Models with these features have been a useful lens to analyze the most recent bout of global inflation that followed large expansions in government borrowing, a global supply shock in the form of the COVID-19 pandemic, and sharp interest rate movements by central banks around the world. However, this literature has focused almost entirely on representative agent economies.

Our motivation for extending this analysis to economies with heterogeneous agents and incomplete markets is two-fold. First, heterogeneous agent models generate consumption responses to income and interest rates that are consistent with the vast body of micro-economic evidence on the joint dynamics of household income and spending.² This is important because household spending pressure is the key force that shapes inflation and interest rates in equilibrium. Second, household heterogeneity has played an important role in both the drivers and consequences of the current inflationary episode. Governments issued vast quantities of new debt to finance transfers that were targeted to certain groups of households. The ongoing spending pressures that are leading many government to run persistent deficits are also highly targeted. Quantitative heterogeneous agent models are a natural environment to study the implications of interventions such as these, as well as the distributional effects of shocks and subsequent policy responses.

This leads us to start building a bridge between the well-studied representative-agent FTPL and workhorse heterogeneous agent models with incomplete markets in the tradition of [Bewley \(1983, 1987\)](#), [Imrohoroglu \(1989\)](#), [Huggett \(1993\)](#) and [Aiyagari \(1994\)](#). In this paper, we take a first step by focusing on flexible-price economies.³

Working in a heterogeneous agent incomplete markets setting also overcomes a limitation of representative agent FTPL models that makes their application to current macroeconomic conditions problematic. Standard representative agent models require governments to run positive primary surpluses in expectation at all points in time. However, in recent decades the US has consistently run primary deficits, and the fiscal positions of the US and many other developed economies look

¹The FTPL literature, which has its roots in [Sargent and Wallace \(1981\)](#) and builds on [Leeper \(1991\)](#), [Sims \(1994\)](#), [Woodford \(1995\)](#) and [Cochrane \(1998\)](#) is too vast to cite in full. See the handbook chapter by [Leeper and Leith \(2016\)](#) and book by [Cochrane \(2023\)](#) for a synthesis of the reach of FTPL models.

²See for example the review article by [Kaplan and Violante \(2022\)](#).

³In ongoing work we extend the analysis to economies with nominal rigidities. See [Kaplan et al. \(2023\)](#).

unlikely to return to surpluses anytime soon.⁴ Heterogeneous agent versions of these models offer a natural setting in which to study price level dynamics with persistent primary deficits. In these versions, the real return on government debt r is less than the growth rate of the economy g , which is also a feature of recent macroeconomic conditions.

Theoretical Analysis. We start by analyzing an endowment economy in which the government runs positive primary surpluses and $r > g$. In these situations, the conditions on monetary and fiscal policy for the price level and inflation to be uniquely determined are essentially unchanged from corresponding representative agent economies. However, there are important quantitative differences that reflect the role of precautionary savings. Unlike in the representative agent economy, changes in fiscal policy in the heterogeneous agent economy lead to movements in the real interest rate. This is because a change in either the level of debt, or the size and distribution of surpluses alters the overall demand for savings among households. For a given setting of monetary policy, these different real rate dynamics imply different paths of inflation. It also means that there are non-trivial inflation dynamics following a one-time fiscal helicopter drop and that the path of inflation depends on the targeting of the fiscal helicopter drop. We use a modified representative agent model with bonds in the utility function to provide intuition for some of these forces.

We then analyze the same heterogeneous-agent economy but with a government that runs a constant primary deficit and $r < g$. We show that, as long as the level of deficits is not too large, equilibria of the heterogeneous agent economy exist. The maximum possible level of deficits is decreasing in the amount of redistribution implicit in the tax and transfer system: more redistribution reduces aggregate precautionary saving and increases real interest payments on debt. For lower levels of deficits, there are generically two steady-state equilibria. This implies that, without additional assumptions, standard FTPL arguments do not uniquely pin down the price level or the path of inflation. The steady-state equilibria are Pareto ranked, with the high debt, high interest rate, low inflation steady-state delivering higher levels of welfare for every household. The low inflation steady-state is saddle-path stable: there is a unique initial price level and subsequent path of inflation and real rates leading to that steady-state. The high inflation steady-state is locally stable: there is a continuum of initial price levels that support paths of inflation and real rates leading to that steady-state.

We discuss various extensions that deliver a unique prediction for the price level and inflation. First, we propose modifications to the model that eliminate the high inflation steady-state altogether, leaving only a unique saddle-path stable steady-state. These modifications include (i) fiscal reaction rules that allow the level of surpluses to respond to deviations of real debt or the real rate from steady-state; and (ii) the introduction of a foreign sector with a relatively inelastic demand for domestic government debt. Second, we propose a policy environment in which the central bank successfully coordinates private sector expectations about long-run inflation. By anchoring long-run

⁴With the exception of 1998-2001, the US has not run a primary surplus since 1970. Moreover, the May 2023 10-year budget projections of the Congressional Budget Office (CBO) estimate that deficits will remain negative at least until 2033: <https://www.cbo.gov/data/budget-economic-data>

inflation expectations to be consistent with the saddle-path stable steady-state, uniqueness is also achieved in the short run, because all the equilibria that converge to the high inflation steady-state are eliminated. With uniqueness of equilibria in hand, we move to the quantitative analysis.

Quantitative Policy Messages. In the quantitative part of the paper, we conduct a series of experiments to illustrate lessons for policy that emerge in the heterogeneous agent setting but are concealed in more traditional representative agent FTPL environments. We use a liquid wealth calibration that targets a steady-state primary deficits of 3.3% of GDP and a debt to annual GDP ratio of 110%, consistent with US averages in the years leading up to the COVID-19 pandemic. This yields a steady-state real interest rate of -1% .

First, we consider the effects of permanently increasing deficits. We calculate that if the government were to permanently increase lump sum transfers to households without raising taxes, the largest sustainable primary deficit would be 4.6% of GDP, or 40% higher than current levels. However the maximum sustainable deficit depends on the degree of social insurance: expanding deficits in a more progressive manner than through lump-sum transfers imply lower maximum deficits. The reason is that tax systems that provide more social insurance weaken the overall precautionary savings motive, thus lowering household demand for government debt. More progressive tax systems therefore reduce fiscal space.

A permanently higher deficit is associated with a lower steady-state real interest rate and less real government debt, as well as a higher long-run inflation rate for a given setting of monetary policy. This is because a larger deficit must be funded by larger real interest receipts, which requires a more negative real rate. With a lower return, households are willing to hold less government debt. The heterogeneous agent framework thus offers an alternative interpretation of discussions around secular stagnation by highlighting the connection between a rising primary deficit, falling real rates and rising inflation.

Next, we study the effects of issuing new debt while holding primary deficits constant: a fiscal helicopter drop. We consider a helicopter drop of around 16% of annual GDP, roughly the size of the fiscal expansion in the US over the course of the COVID-19 pandemic. Consistent with the representative agent experiments in [Cochrane \(2022\)](#), we find that this generates an immediate jump in the price level. However, relative to the representative agent benchmark, in our economy there is an additional 30% initial increase in the price level. This amplification is driven by redistribution and MPC heterogeneity: the downward revaluation of nominal debt that restores equilibrium in the representative agent economy entails large amounts of redistribution from wealthy to poor households in the heterogeneous agent economy. This reallocation of wealth generates upward pressure on consumption, which increases real rates and interest payments on government debt, thereby causing a larger initial jump in the price level. A targeted helicopter drop such as that implemented in the US, which explicitly targets high MPC households, generates further additional short-term inflationary pressures. Lastly, we study the effects of purely redistributive policies that hold both debt and deficits constant, and show that budget neutral redistribution is inflationary. We illustrate these effects by way of numerical experiments in which the government levies a one-

time wealth tax on household in the top percentiles of the wealth distribution, and redistributes the proceeds lump-sum to households in the bottom half of the wealth distribution. As with the fiscal helicopter drop, real redistribution towards high MPC households places upward pressure on consumption, which in equilibrium leads to a temporarily higher real interest rate and a downward revaluation of real assets through a jump in the price level.

Related Literature. This paper is part of a small but growing literature that explores the FTPL in models with incomplete markets. [Bassetto and Cui \(2018\)](#) show that a model of overlapping generations and a model in which government debt provides special liquidity services can give rise to multiple steady-states in which the real interest rate on government debt is below the growth rate of output. They emphasize that the FTPL can fail to deliver price level determinacy in these settings. [Brunnermeier et al. \(2020\)](#) and [Miao and Su \(2021\)](#) consider models with idiosyncratic return risk and study fiscal rules that can deliver price level uniqueness in low interest rate environments.

Our work differs from these papers in three respects. First, we consider the implications of the FTPL in a [Bewley \(1987\)](#) economy in which market incompleteness arises from uninsurable income risk. In doing so, we emphasize the importance of MPC heterogeneity in driving price level and inflation dynamics. Second, we explore a wide class of fiscal, monetary, and institutional specifications and show how they deliver price level uniqueness in models in which the government runs persistent primary deficits. Third, we quantitatively explore the response of economic aggregates to unanticipated shocks in flexible price, low-interest rate economies with persistent deficits. To the best of our knowledge, the messages we deliver about the role of precautionary savings and MPC heterogeneity in driving price level, inflation and real rate dynamics have not been discussed in previous work.⁵

Our work also relates to the literature that studies the implications of low interest rate environments for government borrowing ([Blanchard, 1985, 2019](#); [Aguilar et al., 2021](#); [Mehrotra and Sergeyev, 2021](#); [Mian et al., 2021a](#); [Reis, 2021](#); [Cochrane, 2021](#); [Kocherlakota, 2023](#)). This work emphasizes that the government can roll over debt indefinitely when the real interest rate on government debt is below the growth rate of the economy. We show that this insight is correct, but only up to a limit: there is a finite upper bound on primary deficits for there to exist an equilibrium in which government debt is valued. We quantify this bound in our calibrated model for the U.S. economy and explore how it depends on the overall level of idiosyncratic income risk and on the level of fiscal redistribution. More generally, we show that constant primary deficits give rise to two different steady-states for government debt and the real interest rate, which can be Pareto ranked.

Finally, our work highlights the importance of household heterogeneity in determining interest rates and inflation. As such, it relates to work that explores the distributional consequences of monetary policy and inflation ([Doepke and Schneider, 2006](#); [Coibion et al., 2017](#); [McKay and Wolf, 2023](#)) and the role of agent heterogeneity in amplifying economic outcomes ([Auclert et al., 2018](#);

⁵Some qualitative aspects of our analysis, such as equilibrium multiplicity with deficits share features with certain monetarist economies. See for example the presentation of Bewley models in Chapter 18 of [Ljungqvist and Sargent \(2018\)](#).

Kaplan et al., 2018; Auclert, 2019). In particular, we show that unanticipated changes in the price level can give rise to non-trivial, persistent dynamics in the real interest rate and inflation due to heterogeneous wealth effects across the distribution.

2 Model Environment

2.1 Households

Demographics. Time is continuous and is indexed by $t \geq 0$. The economy is populated by a continuum of households indexed by $j \in [0, 1]$.

Endowments. Real aggregate output y_t is exogenous and grows at a constant rate $g \geq 0$. Household j receives a stochastic idiosyncratic share z_{jt} of aggregate output. The shares z_{jt} are independent across households and a law of large numbers holds so that there is no economy-wide uncertainty,

$$\int_{j \in [0,1]} z_{jt} dj = 1 \text{ for all } t \geq 0. \quad (1)$$

In our baseline model we assume that z_{jt} follows an N -state Poisson process with switching intensities $\lambda_{z,z'}$. The lowest value of the endowment share $\underline{z}$ is strictly positive, $\underline{z} > 0$, from which it follows that the natural borrowing limit is below zero.⁶ In Appendix B.1 we describe a version of the model in which z_{jt} follows a diffusion process.

Assets. Households trade a short-term risk-free bond that yields a nominal flow return i_t . We denote the nominal bond holdings of household j at time t by A_{jt} . These short-term bonds are the unit of account in the economy and we let P_t denote the price of output in terms of this short-term bond.

Preferences. Households take the path of aggregate variables $\{P_t, i_t, y_t\}_{t \geq 0}$ as given and choose real consumption flows $\tilde{c}_{jt}$ to maximize

$$\mathbb{E}_0 \int e^{-\tilde{\rho}t} \frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma} dt. \quad (2)$$

where the expectation is taken over the idiosyncratic endowment process z_{jt} . We denote the household's discount rate by $\tilde{\rho} > 0$.

Nominal Household Budget Constraint. Initial nominal assets A_{j0} are given. For $t > 0$, households face a flow budget constraint

$$dA_{jt} = [i_t A_{jt} + (z_{jt} - \tau_t(z_{jt})) P_t y_t - P_t \tilde{c}_{jt}] dt. \quad (3)$$

⁶In our quantitative experiments in which we allow for borrowing, the interest rate on borrowing is always positive so the natural borrowing limit is well-defined.

The path of tax and transfer functions $\tau_t(z)$ is set by the fiscal authority and is described in more detail below. Nominal savings dA_{jt} are equal to the sum of asset income $i_t A_{jt}$ and endowment income net of taxes and transfers $(z_{jt} - \tau_t(z_{jt})) P_t y_t$, minus consumption expenditures $P_t \tilde{c}_{jt}$. In our baseline model we assume that households cannot borrow $A_{jt} \geq 0$, but we allow for borrowing in our quantitative analysis in Section 5. Appendix B.2 contains an analysis of the model with borrowing.

Price Level and Inflation. Since this is a flexible-price economy, the price level P_t may exhibit jumps. For ease of notation and exposition, in the main body of the paper we restrict the price level to jump only at $t = 0$, after which it follows a deterministic path. Since there is no intrinsic (i.e., fundamental) aggregate uncertainty, this implies perfect foresight over aggregate variables for $t > 0$.⁷ For $t > 0$, we define the inflation rate by

$$\frac{dP_t}{P_t} = \pi_t dt. \quad (4)$$

De-trended Real Household Budget Constraint. We denote de-trended real assets and de-trended consumption as

$$a_{jt} := \frac{A_{jt}}{P_t y_0 e^{gt}} \quad c_{jt} := \frac{\tilde{c}_{jt}}{y_0 e^{gt}} \quad (5)$$

For $t > 0$, we can re-write the nominal household budget constraint (3) in de-trended real terms:

$$da_{jt} = [r_t a_{jt} + z_{jt} - \tau_t(z_{jt}) - c_{jt}] dt \quad (6)$$

where

$$r_t := i_t - \pi_t - g \quad (7)$$

is the growth-adjusted real rate. At $t = 0$, de-trended real assets a_{j0} are given by the ratio of initial nominal assets A_{j0} to the endogenous initial price level P_0 .

Relative Asset Holdings. We use A_t and a_t to denote aggregate nominal household assets and aggregate de-trended real household assets, respectively:

$$A_t := \int_{j \in [0,1]} A_{jt} dj \quad a_t := \int_{j \in [0,1]} a_{jt} dj$$

We denote the share of assets held by household j at time t by $\omega_{jt} := \frac{A_{jt}}{A_t} = \frac{a_{jt}}{a_t}$ and note that by construction

$$\int_{j \in [0,1]} \omega_{jt} dj = 1 \text{ for all } t \geq 0. \quad (8)$$

⁷Studying perfect foresight solutions with a single probability-zero jump at time zero is the commonly maintained in FTPL models (Leeper, 1991; Sims, 2011; Cochrane, 2018). Moreover, the absence of any aggregate uncertainty implies that the price level cannot exhibit jumps for $t > 0$ in discrete time, representative agent FTPL models (Cochrane, 2023).

Recursive Formulation of Household Problem. Given paths of real rates r_t and tax functions τ_t , the household problem can be expressed recursively via the Hamilton-Jacobi-Bellman Equation (HJB)

$$\begin{aligned} \rho V_t(a, z) - \partial_t V_t(a, z) &= \max_c \frac{c^{1-\gamma}}{1-\gamma} + \partial_a V_t(a, z) [r_t a + z - \tau_t(z) - c] \\ &\quad + \sum_{z' \neq z} \lambda_{z, z'} [V_t(a, z') - V_t(a, z)], \end{aligned} \quad (9)$$

together with the boundary condition $\partial_a V_t(0, z) \geq (z - \tau_t(z))^{-\gamma}$ that ensures that the borrowing constraint $a \geq 0$ is satisfied. The growth-adjusted discount rate ρ in (9) is defined as $\rho = \tilde{\rho} - (1-\gamma)g$.

The optimal consumption function $c_t(a, z)$ that solves the HJB is defined by

$$c_t(a, z) = [\partial_a V_t(a, z)]^{-\frac{1}{\gamma}}. \quad (10)$$

The associated savings function is denoted by

$$\varsigma_t(a, z) := r_t a + z - \tau_t(a, z) - c_t(a, z) \quad (11)$$

If a value function $V_t(a, z)$ solves the HJB (9) and also satisfies the boundedness condition

$$\lim_{T \rightarrow \infty} \mathbb{E}_T [e^{-\rho T} V_T(a_{jT}, z_{jT})] = 0, \quad (12)$$

then the stochastic process for consumption defined by (10) solves the sequence version of the household problem (2).⁸

The distribution of households across real asset holdings and endowment shares $g_t(a, z)$ satisfies the Kolmogorov Forward Equation (KFE)

$$\partial_t g_t(a, z) = -\partial_a [g_t(a, z) \varsigma_t(a, z)] - g_t(a, z) \sum_{z' \neq z} \lambda_{z, z'} + \sum_{z' \neq z} \lambda_{z', z} g_t(a, z'). \quad (13)$$

We use $f_t(\omega, z)$ to denote the distribution of households across asset shares and endowment shares. For a given path of aggregate real household assets a_t , the distributions $f_t(\omega, z)$ and $g_t(a, z)$ are related by

$$f_t(\omega, z) = g_t(\omega a_t, z). \quad (14)$$

⁸See Theorem 3.5.3 in [Pham \(2009\)](#). The expectation in (12) is taken with respect to the stochastic process for idiosyncratic income and assets for household j , given by the budget constraint (6).

2.2 Government

Nominal Government Budget Constraint. We assume a fiscal authority that issues short-term nominal government debt B_t subject to the budget constraint:

$$dB_t = [i_t B_t - s_t P_t y_t] dt \quad (15)$$

where s_t is the ratio of primary surpluses to output and is determined by the tax and transfer function as

$$s_t = \int_{j \in [0,1]} \tau_t(z_{jt}) dj \quad (16)$$

Equation (15) defines the evolution of nominal government debt. This is a backward-looking equation where the initial level of nominal government $B_0 > 0$ is given. We restrict $B_t \geq 0$ so that the government can only borrow and not lend.⁹

De-trended Real Government Budget Constraint. We denote de-trended real government debt (or the debt-output ratio) by b_t ,

$$b_t = \frac{B_t}{P_t y_0 e^{gt}}. \quad (17)$$

For $t > 0$, real debt b_t evolves according to the real version of the government budget constraint given by (15):

$$db_t = [r_t b_t - s_t] dt. \quad (18)$$

Real debt increases whenever real interest rate payments exceed real primary surpluses. At $t = 0$, de-trended real debt b_0 is a jump variable given by the ratio of exogenously given initial nominal debt B_0 to the endogenous initial price level P_0 .

Fiscal Policy. For our baseline analysis we focus on a time-invariant tax and transfer function $\tau_t(z) = \tau^*(z)$, so that surpluses or deficits are a constant fraction of real output $s_t = s^*$. In Section 4.3, we generalize the analysis to allow for a broader class of fiscal rules of the form

$$s_t = s(b_t, r_t). \quad (19)$$

These rules allow primary surpluses to respond to real aggregate debt, real interest rates or real interest payments and play an important role in determining the price level when governments run persistent deficits, $s_t < 0$.

Monetary Policy. For our baseline analysis we focus on a nominal interest rate peg $i_t = i^*$. In our quantitative analysis in Section 5 we allow for long-term debt and a richer class of Taylor-type rules for nominal interest rates. We also discuss how allowing for other monetary rules affects our results about the determination of the price level and inflation in Section 2.3.

⁹Introducing government consumption would be subsumed in s_t in Equation (15), thereby leaving the key mechanisms of our model unchanged.

2.3 Equilibrium

We first define a *real equilibrium* as a collection of real variables which satisfy household optimality, are consistent with their respective law of motions, and obey market clearing.

Definition 1. *Given (i) a constant tax and transfer function $\tau^*(z)$; and (ii) an initial distribution of households across asset and endowment shares $f_0(\omega, z)$, a real equilibrium is a collection of variables:*

$$\{V_t(a, z), c_t(a, z), f_t(\omega, z), g_t(\omega a_t, z), a_t, b_t, r_t\}_{t \geq 0} \quad (20)$$

such that, for all $t \geq 0$:

1. the value function $V_t(a, z)$ solves the HJB (9) and satisfies the boundedness condition (12)
2. the consumption function is defined by (10)
3. the distribution of asset levels $g_t(\omega a_t, z)$ solve the KFE (13)
4. the distribution of household endowment shares is consistent with aggregate assets

$$f_t(\omega, z) = g_t(\omega a_t, z) \quad (21)$$

5. the aggregate holding of real household assets a_t equals the mean of the distribution $g_t(\omega a_t, z)$
6. the path of government debt b_t satisfies the government budget constraint (18)
7. the asset market clears $a_t = b_t$

Note that by Walras' law, asset market clearing implies that the goods market clearing condition is also satisfied:

$$\int_{j \in [0,1]} c_{jt} dj = 1 \text{ for all } t \geq 0$$

Price Level and Inflation Determination. Under our assumptions about monetary and fiscal policy, each real equilibrium implies a unique initial price level P_0 and a subsequent unique path of inflation π_t . These are determined as follows. Each real equilibrium contains an initial value of real government debt b_0 . Since initial nominal debt B_0 is given, the initial price level is determined as

$$P_0 = \frac{B_0}{b_0}.$$

The path of inflation is uniquely determined by the equilibrium path of real rates r_t and the nominal rate i^* which is set by the monetary authority as

$$\pi_t = i^* - r_t - g.$$

Each real equilibrium therefore implies a unique path for the price level P_t for all $t \geq 0$.

It follows that uniqueness of the price level and inflation is equivalent to uniqueness of a real equilibrium. If there is more than one real equilibrium then there will be more than one possible path for the price level. But if the real equilibrium is unique, then there is only one possible path for the price level, which is determined by initial nominal debt and monetary policy. As a result, we

focus most of our analysis on the existence and uniqueness of real equilibria, with the understanding that whenever the real equilibrium is unique, so too is the price level and inflation.

Monetary Policy Rules. With flexible prices, the equivalence between uniqueness of real equilibria and uniqueness of the path of prices does not depend on our assumption of a nominal interest rate peg $i_t = i^*$. If the monetary authority instead follows an instantaneous feedback Taylor Rule of the form

$$i_t = i^* + \phi_m(\pi_t - \pi^*) \quad (22)$$

then inflation is uniquely determined as

$$\pi_t = \frac{i^* - \phi_m \pi^* - r_t - g}{1 - \phi_m}$$

If the monetary authority follows a lagged feedback Taylor Rule of the form

$$di_t = -\theta_m [i_t - i^* - \phi_m(\pi_t - \pi^*)] dt \quad (23)$$

then i_0 cannot jump and is inherited from the past. Initial inflation is determined as $\pi_0 = i_0 - r_0 - g$ and subsequent inflation is determined as the unique forward solution to the ordinary differential equation

$$d\pi_t = -\theta_m [\pi_t - \phi_m(\pi_t - \pi^*) + r_t - (g - i^*)] dt - dr_t.$$

Depending on parameter configurations, prices and inflation may not remain bounded, but there is nothing in the equilibrium definition that rules out such paths.

3 Primary Surpluses $s^* > 0$

We start by studying equilibria when the fiscal authority runs positive primary surpluses. First we analyze steady-state equilibria. We then analyze non-stationary equilibria. This analysis leads us to argue that there is a unique real equilibrium and hence the price level is uniquely determined at all points in time. We use the examples of a permanent reduction in surpluses and a one-time fiscal helicopter drop to illustrate the different dynamics of inflation and real rates in the heterogeneous agent economy compared to its representative agent counterpart.

3.1 Stationary Equilibrium

Household Asset Demand. In a stationary equilibrium, the real rate r_t is constant. Under regularity conditions that are well understood, with a constant interest rate r and transfer function $\tau(z)$, the solution to (9) and (13) implies a unique stationary distribution $g(a, z; r)$.¹⁰ We use this result to construct a function $\mathbf{a}(r)$ that maps different interest rates into the aggregate quantity of

¹⁰See e.g. Bewley (1995), Stokey et al. (1989), and Aiyagari (1994).

real assets held by households in the corresponding stationary distribution,

$$\mathbf{a}(r) := \int_{a,z} ag(a, z; r) da dz$$

It is well known that $\lim_{r \rightarrow \rho} \mathbf{a}(r) = \infty$. In addition we will assume that the function $\mathbf{a}(r)$ is continuous, differentiable and strictly increasing.¹¹ In Appendix A.1 we show that there exists an interest rate $\underline{r} < 0$ below which households do not hold any assets in the stationary distribution, so that $\mathbf{a}(r) = 0$ for all $r \leq \underline{r}$. The blue line in Figure 1 labelled $\mathbf{a}(r)$ is an example of a typical stationary asset demand function.

Government Asset Supply. In a stationary equilibrium, the government budget constraint defines a steady-state asset supply function $\mathbf{b}(r)$. This is obtained by setting $db_t = 0$ in (15),

$$\mathbf{b}(r) = \frac{s}{r}. \quad (24)$$

Since $b_t \geq 0$, this supply function takes the shape of a downward-sloping hyperbola in the positive quadrant as illustrated by the red line labelled $\mathbf{b}(r)$ in Figure 1.

Stationary Equilibrium. A stationary equilibrium requires that

$$\mathbf{a}(r) = \mathbf{b}(r)$$

so that the asset market clears. Given our assumptions, there is a unique stationary real equilibrium shown as (b^*, r^*) in Figure 1. The assumption that primary surpluses are positive $s^* > 0$ implies that the stationary equilibrium real rate r^* is positive.

The unique stationary equilibrium in the corresponding representative agent economy is shown by the point (b^{RA}, r^{RA}) in Figure 1. In the representative agent economy the household asset demand curve is perfectly elastic at $r = \rho$. As is well known, in the heterogeneous agent economy the real rate is lower and the level of real government debt is higher than in the representative agent economy.

3.2 Non-Stationary Equilibrium

Because there is a unique stationary real equilibrium, in order to pin down the price level and inflation it suffices to rule out multiplicity of non-stationary real equilibria. Before tackling this in the heterogeneous agent economy, it is useful to recap the argument in the representative agent economy.

Uniqueness in Representative Agent Economies. In a representative agent economy, consumption satisfies an Euler equation of the form

$$\frac{dc_t}{c_t} = \frac{1}{\gamma} (r_t - \rho) dt$$

¹¹ Achdou et al. (2022) show that sufficient conditions for this to be true are $\gamma \leq 1$ and $\underline{a} \geq 0$.

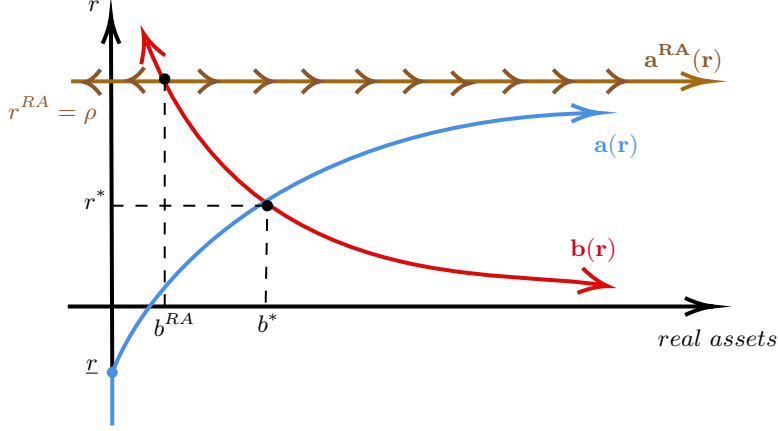


Figure 1: Steady state equilibrium with positive surpluses

In an endowment economy, goods market clearing implies $dc_t = 0$ and hence in equilibrium $r_t = \rho$ at all points in time, not just in a stationary equilibrium. Graphically, this means that the economy lives on the brown horizontal line labelled $\mathbf{a}^{RA}(r)$ in Figure 1 at all points in time. The real government budget constraint implies that $db_t = [\rho b_t - s^*]dt$. It follows that real debt is increasing when it is above steady-state, and decreasing when it is below steady-state, as illustrated by the arrows in Figure 1. Paths in which debt is increasing are ruled out as equilibria by showing that they violate a transversality condition that is a necessary condition for household optimality. Paths in which debt is decreasing are ruled out as equilibria since they violate the household's borrowing constraint in finite time and would imply that the government is a net lender. This argument is formalized in Appendix C. It follows that the stationary equilibrium is the unique real equilibrium and hence the initial price level and subsequent inflation are uniquely determined:

$$\begin{aligned} P_0 &= \frac{B_0}{b^{RA}} \\ \pi_t^{RA} &= i^* - \rho - g \end{aligned}$$

Equilibrium paths display an initial jump in the price level at $t = 0$, and a constant inflation rate equal to steady-state inflation for $t > 0$.

Uniqueness in Representative Agent Economies with Bonds-In-Utility. The heterogeneous agent economy differs from the representative agent economy in part because the steady-state asset demand function is not perfectly elastic. In Appendix D we describe a simple representative agent economy in which households directly generate utility by holding real government debt. This economy features a steady-state asset demand function $\mathbf{a}^{BIU}(r)$ that has the same qualitative properties as $\mathbf{a}(r)$. In this economy, all equilibria lie on the one-dimensional manifold $\mathbf{a}^{BIU}(r)$ at all points in time, and away from steady-state the dynamics of government debt are unstable. A transversality condition and borrowing constraint rule out explosive paths in either direction as

equilibria and hence the steady-state equilibrium is the unique equilibrium. The initial price level and subsequent inflation are uniquely determined. With positive primary surpluses, the difference between this economy and the standard representative agent economy is that the real interest rate is endogenous and depends on the level of surpluses. See Appendix D.3 for a formal argument.

State-Space Representation for Heterogeneous Agent Economy. Establishing that there is no multiplicity of non-stationary equilibria in the heterogeneous agent economy is more difficult than in the representative agent bonds-in-utility economy because the equilibria do not lie on a one-dimensional manifold. The aggregate state for the heterogeneous agent economy consists of the household asset and endowment distribution $g_t(a, z)$.¹² It is useful to partition this distribution into two components, which we denote by $\Omega_t := \{f_t(\omega, z), b_t\}$

- (i) $f_t(\omega, z)$: the joint distribution of household asset shares and endowment shares
- (ii) b_t : the level of real government debt.

The reason for partitioning the aggregate state in this way is that the two components have different dynamic properties. The distribution $f_t(\omega, z)$ is backward-looking and cannot jump. The level of real debt is a jump variable. It can jump because different values of the initial price level P_0 revalue the outstanding stock of nominal bonds B_0 . Partitioning in this way makes it clear that although the household distribution $g_0(a, z)$ can jump, it can only jump along a single dimension such that the relative wealth holdings of each household remains unchanged. Using this state variable, we can write the consumption function $c_t(a, z)$ as $c(a, z, \Omega_t)$, where dependence on time is completely subsumed in the aggregate state.

Roadmap. Our discussion of uniqueness involves two steps. First, we show that any paths for b_t that diverge in either direction are not consistent with equilibrium because they evolve even-
tual violation of either the borrowing constraint or a necessary household transversality condition. Second, we argue that the dynamics of Ω_t around the unique stationary equilibrium are locally saddle-path stable. Given an initial distribution $f_0(\omega, z)$ in the vicinity of $f^*(\omega, z)$, saddle-path stability implies that there is a unique initial value for the jump variable b_0 and unique subsequent paths of the aggregate state Ω_t such that the economy converges to $\Omega^* = \{f^*(\omega, z), b^*\}$.¹³

Ruling Out Explosive Equilibria. In Appendix A.3, we show that all paths of government debt b_t that grow at rate $r_t < \rho$ imply eventual violation of the following household transversality condition:

$$\lim_{T \rightarrow \infty} \mathbb{E}_{jt} \left[e^{-\rho T} c_T(a_{jT}, z_{jT})^{-\gamma} a_{jT} \right] \leq 0. \quad (25)$$

¹²The absence of the interest rate r_t from the aggregate state is not immediately obvious. However, as we verify below, in equilibrium it is implied by the joint distribution $g_t(a, z)$. In our quantitative analysis, we consider unanticipated time-varying shocks to various exogenous parameters. In these cases, the state space Ω_t needs to be expanded to include the law of motion for these exogenous driving processes.

¹³We must also rule out the possibility of non-stationary equilibria that remain bounded away from the stationary steady-state and involve cycles or similar dynamics. Although we cannot prove that no such equilibria exist, we have not encountered any numerically.

and hence cannot be part of equilibrium.¹⁴ Sufficient conditions for the equilibrium interest rate r_t in the heterogeneous agent economy to be below the discount rate ρ for all $t \geq 0$ are established in Appendix B.1.

Useful Characterization of Equilibrium Real Rate. In Appendix A.1 we derive expressions for expected consumption growth $\mathbb{E}_t[dc_{jt}]$ for constrained and unconstrained households. Here we use the short-hand notation $c_{jt} := c(a_{jt}, z_{jt}, \Omega_t)$ to denote the consumption of household j at time t . By aggregating these expressions across households, applying the law of iterated expectations, and imposing market clearing we derive the following relationship between the real rate and the aggregate state Ω_t ,

$$0 = \underbrace{\frac{\mathcal{C}_t^u}{\gamma}(r_t - \rho)}_{\text{intertemporal substitution}} + \underbrace{\frac{\mathcal{C}_t^u}{\gamma} \tilde{\mathbb{E}}_t^u \left[\sum_{z'} \lambda_{z_j, z'} \left(\frac{c(\omega_j, z', \Omega_t)}{c_{jt}} \right)^{-\gamma} \right]}_{\text{precautionary motive}} + \underbrace{\mathbb{E}_t \left[\sum_{z'} \lambda_{z_j, z'} \{c(\omega_j, z', \Omega_t) - c_{jt}\} \right]}_{\text{intertemporal smoothing}} \quad (26)$$

The expectation operator $\tilde{\mathbb{E}}_t^u$ is a consumption-weighted mean across the set of unconstrained households, and $\mathcal{C}_t^u$ is the total consumption of unconstrained agents. Appendix A.2 contains a full derivation of this relationship.¹⁵

Equation (26) can be interpreted as balancing three forces driving changes in aggregate consumption that must net out to zero in an endowment economy. The first term is an intertemporal substitution motive for saving. The second term is the average precautionary savings motive. The presence of $\mathcal{C}_t^u$ captures the fact that this saving motive is only active for unconstrained households. The final term reflects an intertemporal motive for smoothing income shocks. In equilibrium, the interest rate is set so that the negative intertemporal substitution motive exactly offsets the combined effects of the precautionary saving and intertemporal smoothing motives.¹⁶

Equation (26) also confirms that the real rate is not required as a separate component of the aggregate state since that equation implicitly defines a time-invariant mapping from Ω_t to r_t that holds at all times in equilibrium. We denote this functional by

$$r_t = \mathbf{r}[\Omega_t]. \quad (27)$$

Local Saddle Path Stability. We derive the dynamics of the the aggregate state Ω_t by expressing the Kolmogorov Forward Equation (13) in terms of asset shares, and substituting the real rate

¹⁴Establishing the transversality condition (25) as a necessary condition for household optimality is non-trivial. Kamihigashi (2001) shows that it is necessary in an analogous deterministic economy. Kamihigashi (2003) shows necessity in a discrete time stochastic economy.

¹⁵Appendix B.1 contains the analogous formula for the real rate functional when idiosyncratic endowments follow a diffusion process.

¹⁶In the special case with quadratic utility, no borrowing constraints (hence, no precautionary saving) and $r_t = \rho$, equation (26) states that consumption is a martingale.

functional (26) into the government budget constraint (18):

$$\partial_t f_t(\omega, z) = -\partial_\omega \left[f_t(\omega, z) \frac{1}{b_t} \{z - \tau^*(z) - c(\omega b_t, z, \Omega_t) + s^* \omega\} \right] \quad (28)$$

$$-f_t(\omega, z) \sum_{z' \neq z} \lambda_{zz'} + \sum_{z' \neq z} \lambda_{z'z} f_t(\omega, z')$$

$$\frac{db_t}{dt} = \mathbf{r}[\Omega_t] b_t - s^* \quad (29)$$

Since this system is comprised of a one-dimensional jump component b_t and an infinite dimensional backward looking component $f_t(\omega, z)$, local saddle-path stability requires that, around the steady-state, this PDE system has one positive eigenvalue and non-positive remaining eigenvalues.

Discretized Economy. Although we are not able to prove saddle-path stability for the full continuum economy, we have found the system to be saddle path stable in our numerical explorations of discretized versions of this economy. Here we offer some intuition for local saddle-path stability from this discretized economy.

We consider a discrete approximation to $f(\omega, z)$ on a grid for relative asset shares of length N_ω , which we denote by the $N \times 1$ vector $\mathbf{f}$ where $N = N_\omega \times N_z$. In Appendix A.4 we show that the finite difference approximation the PDE system (28) is given by the system of $N + 1$ ODEs

$$\frac{d\mathbf{f}}{dt} = \mathbf{A}_\omega [\mathbf{f}_t, b_t]^T \mathbf{f}_t + \mathbf{A}_z^T \mathbf{f}_t \quad (30)$$

$$\frac{db}{dt} = \mathbf{r}[\mathbf{f}_t, b_t] b_t - s^* \quad (31)$$

The matrices $\mathbf{A}_\omega [\mathbf{f}, b]^T$ and $\mathbf{A}_z^T$ are upwind finite difference approximations to the two linear operators that comprise the KFE for (ω, z) .¹⁷

The dependence of $\mathbf{A}_\omega [\mathbf{f}, b]^T$ on the distribution $\mathbf{f}$ and real debt b arises for three reasons. First, a change in aggregate wealth b has a common effect on the interest earnings at all points in the wealth distribution. This direct effect is reflected by the b_t in the denominator of the top line of (28). Second, a change in aggregate wealth change changes consumption of all households via a wealth effect. This is reflected in the b_t in the first argument of the consumption function in (28). Finally, there are further general equilibrium effects on consumption because of future interest rate dynamics. These are reflected in the dependence of the consumption function on the aggregate state Ω_t in the third argument.

In Appendix A.4, we linearize the discretized system (30) around the steady state $(\mathbf{f}^*, b^*)$ and show that the local dynamics are approximately

$$\begin{pmatrix} \frac{d\mathbf{f}}{dt} \\ \frac{db}{dt} \end{pmatrix} \approx \begin{pmatrix} \mathbf{A}_\omega^*{}^T + \mathbf{A}_z^T & \nabla_b \mathbf{A}_\omega^T [\mathbf{f}^*, b^*] \\ 0 & b^* \{ \partial_b \mathbf{r} [\mathbf{f}^*, b^*] - (-\frac{r^*}{b^*}) \} \end{pmatrix} \begin{pmatrix} \mathbf{f}_t - \mathbf{f}^* \\ b_t - b^* \end{pmatrix} \quad (32)$$

¹⁷The transposes reflect the fact that these matrices are constructed by first constructing finite difference approximations to the adjoint operators in (28).

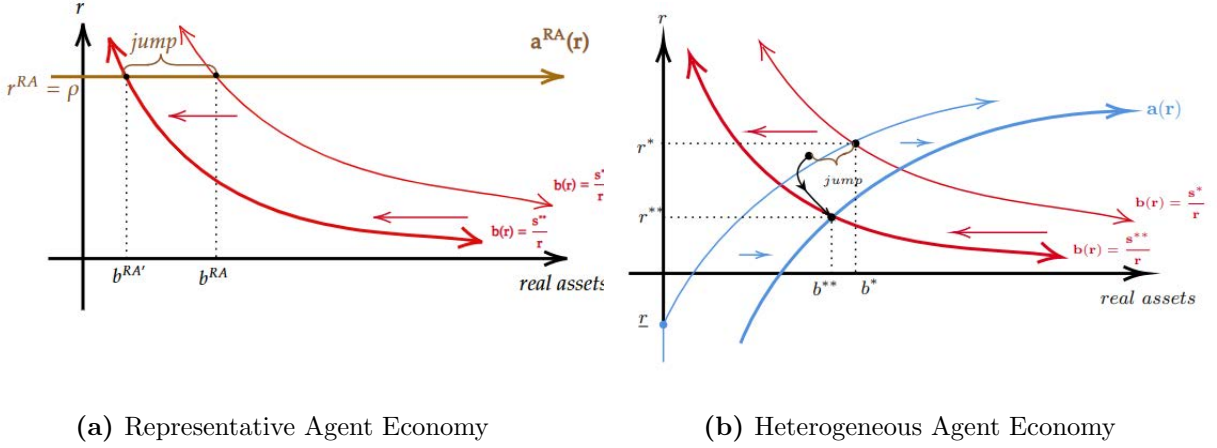


Figure 2: A permanent reduction in surpluses

where term $\nabla_b \mathbf{A}_\omega^T [\mathbf{f}^*, b^*]$ is the $N_\omega \times 1$ vector of derivatives of $\mathbf{A}_\omega^T$ with respect to real debt b .

The approximation in (32) refers to the zero in the bottom left element of the Jacobian. Our approximation requires this term to be small only relative to the term in the bottom right element of the Jacobian. This means we require that around the steady-state, the dynamics of real government debt are much more sensitive to changes in the *level* of real debt, holding the distribution of asset shares constant, than they are to the changes in the *distribution* of asset shares, holding the level of real debt constant.¹⁸

In this case, the Jacobian is approximately block triangular, allowing us to sign the eigenvalues of the full system: $\mathbf{A}_\omega^* + \mathbf{A}_z^T$ is an irreducible transition rate matrix and so has a single zero eigenvalue and remaining negative eigenvalues. The sign of the remaining eigenvalue is given by the sign of $\mathbf{r} [\mathbf{f}^*, b^*] + \partial_b \mathbf{r} [\mathbf{f}^*, b^*] b^*$. The first term is the steady-state interest rate. The second term is the inverse of the derivative of the steady-state household asset demand curve, multiplied by the level of steady-state assets. Both terms are positive under constant positive surpluses. Hence, under our assumptions, the remaining eigenvalue is strictly positive and the economy saddle-path stable whenever $s^* > 0$.

Additional Fiscal Rules. The preceding analysis assumed a fixed surplus s^* . A more general fiscal rule that nests the fixed surplus as a special case is one in which primary surpluses respond to deviations of real interest payments from their steady state level:

$$s_t = s^* + \phi_s (r_t b_t - s^*). \quad (33)$$

¹⁸This assumption might appear at odds with our substantive messages that emphasize changes in the distribution of real wealth as a quantitatively important factor in driving inflation and price level dynamics. However, as our simulations confirm, these are not contradictory: the feedback from the distribution of shares to the debt dynamics are large enough to be quantitatively meaningful, but would need to be orders of magnitude larger to alter the qualitative features of the dynamic system.

The constant surplus policy s^* is a special case of this rule in which $\phi_s = 0$. With this more general fiscal rule the eigenvalue associated with the dynamics of government debt b_t in (32) is given by $(1 - \phi_s)(\mathbf{r}[\mathbf{j}^*, b^*] + \partial_b \mathbf{r}[\mathbf{j}^*, b^*] b^*)$. It follows that as long as $\phi_s < 1$, the sign of this eigenvalue, and hence the qualitative properties of the dynamic system, are unchanged. The price level and inflation remain uniquely determined. This is an example of an active fiscal rule in the language of Leeper (1991).

However, if $\phi_s > 1$, the eigenvalue associated with government debt b_t is negative. In this case, the equilibrium dynamics around the steady-state are locally stable and so neither the initial level of real debt b_0 , price level or inflation are uniquely determined. This is an example of a passive fiscal rule in the language of Leeper (1991).¹⁹

3.3 Examples

Permanent Reduction in Surpluses. We use a permanent reduction in surpluses as an example to illustrate the saddle-path dynamics. Consider a fiscal authority that unexpectedly changes the tax and transfer function from $\tau^*(z)$ to $\tau^{**}(z) = (1 - \Delta_s)\tau^*(z)$ so that primary surpluses decline to $s^{**} = (1 - \Delta_s)s^*$, with $\Delta_s \in (0, 1)$. The new steady-state government bond supply function is

$$\mathbf{b}(r) = \frac{s^{**}}{r}$$

which is displayed as a leftward shift of the red line in Figure 2.

First, consider the effects of this change in the representative agent economy. The initial steady-state equilibrium before the change is indicated by b^{RA} . When the level of surpluses fall, the economy immediately jumps to the new steady-state equilibrium at the point labelled $b^{RA'}$. The level of real debt immediately falls to $(1 - \Delta_s)b^{RA}$, which is achieved by a one-time upward jump in the price level from P_0 to $\frac{P_0}{1 - \Delta_s}$ with no change in either the real interest rate or inflation. The stock of nominal debt is unchanged, but real surpluses are reduced and thus the price level must jump to lower the real value of outstanding debt.

In the heterogeneous agent economy, the initial steady-state equilibrium is indicated by the point (b^*, r^*) . In general, a change in the tax and transfer function induces a shift in the steady-state household asset demand function for two reasons: (i) it affects disposable income; and (ii) it alters the degree of risk-sharing in the economy. In this example, the effect is to shift the $\mathbf{a}(r)$ curve to the right. The new steady-state after the change is indicated by the point (b^{**}, r^{**}) . Unlike in the representative agent economy, the economy does not jump immediately to the new steady-state. Rather, saddle-path dynamics imply that on impact of the change there is a one-time jump in the level of real debt to the unique value of b_0 that is consistent with non-explosive dynamics, which then determines a unique r_0 through the real rate functional (26). This is indicated by the leftward jump in Figure 2b. The initial jump is achieved by a rise in the price level that devalues

¹⁹When $\phi_s = 1$, fiscal policy is strongly passive in the sense that the government holds real debt b_t constant and adjusts surpluses in responses to changes in the real rate as $s_t = r_t b^*$. This is the case considered in the experiments in Kaplan et al. (2018).

all households' wealth proportionately.²⁰ This shift in the wealth distribution then induces trading among households as the interest rate falls smoothly to its new steady-state level. Without any change in monetary policy, inflation rises smoothly during this transition until it reaches its new steady-state level, which is higher than in the original steady state by the amount $r^{**} - r^*$.

One-Time Fiscal Helicopter Drop. The different inflation dynamics between the representative agent and heterogeneous agent economies is illustrated even more starkly by a fiscal helicopter drop. Assume that the economy is initially in steady-state, when the fiscal authority unexpectedly issues an amount Δ_B of new nominal debt, which it distributes lump-sum to households. Since there is no change in future surpluses, in the representative agent economy the only effect of the helicopter drop is to cause a one-time jump in the price level from P_0 to $P_0(1 + \frac{\Delta_B}{B_0})$, leaving the real value of debt unchanged with no movement in inflation.

However, in the heterogeneous agent economy, the fiscal helicopter drop induces wealth redistribution which leads to movements in the equilibrium real rate and inflation. To understand the dynamics, assume that the price level did in fact jump to $P_0(1 + \frac{\Delta_B}{B_0})$ as in the representative agent economy, leaving aggregate real debt unchanged. Because the nominal transfer is lump-sum to all households, but the revaluation from the change in the price level is in proportion to each households' assets, there is a redistribution of real wealth from high wealth to low-wealth households, even though aggregate real debt is unchanged. This puts upward pressure on consumption because marginal propensities to consume are bigger for low wealth households. Current and future real interest rates must rise in order to offset this additional demand for consumption. In turn, the higher interest payments on debt imply that the initial level of real government debt must decrease for the economy to converge back to steady-state. This necessitates a larger jump in the price level relative to the representative agent economy. We explore these effects in more detail in our quantitative experiments in Section 5.

4 Primary Deficits $s^* < 0$

We now assume that the tax and transfer function $\tau^*(z)$ is such that $s^* < 0$ so that the fiscal authority runs a constant primary deficit. We first show that generically there are zero or two steady-state equilibria, depending on the level of deficits. We then characterize the out of steady-state dynamics and non-stationary real equilibria. We end this section with a discussion of alternative ways to restore uniqueness of a saddle-path stable equilibrium and hence a unique path for prices.²¹ The steady-state real interest rate and real assets are $r^* = 0$ and $b^* = \mathbf{a}(0)$, respectively.

²⁰In general, the initial jump in the price level may undershoot or overshoot its long-run value depending on the nature of the transfer function.

²¹In Appendix A.5 we consider the case where $s^*=0$. Like in the case with $s^* > 0$, there is a unique equilibrium with a finite price level and the path of prices is uniquely determined.

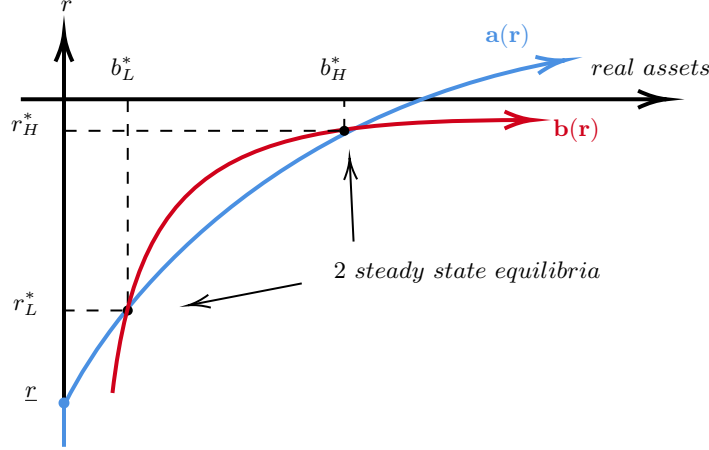


Figure 3: Steady-state equilibria with deficits

4.1 Stationary Equilibria

Household Asset Demand. With $s^* < 0$, the steady-state household asset demand $\mathbf{a}(r)$ function is qualitatively unchanged. It is displayed by the blue line in Figure 3.

Government Asset Supply. The steady-state of the real government budget constraint (18) takes the same form with $s^* < 0$ as it does with $s^* > 0$,

$$\mathbf{b}(r) = \frac{s^*}{r}.$$

Since an equilibrium requires that $b_t \geq 0$, so that the government borrows and households save, the steady-state government budget constraint is an upward-sloping hyperbola as displayed by the red line labelled $\mathbf{b}(r)$ in Figure 3.

Stationary Equilibria. With $s^* < 0$, any steady-state equilibria must have a real rate that is below the growth rate of the economy $r^* = i^* - \pi^* - g < 0$. From Figure 3, it is immediate that if such a steady-state equilibrium exists, then generically there will be two steady-state equilibria, as indicated by the two intersections of the asset supply and demand curves.²²

In these equilibria, households make real interest payments to the government, which the government uses to finance a constant stream of deficits. Households are willing to pay for the privilege of lending to the government because of the precautionary savings motive. Accumulating claims against the government is the only way to self-insure against idiosyncratic endowment shocks.

For a given nominal interest rate, the top equilibrium (labelled (b_H^*, r_H^*)) has a higher level of real debt, higher real interest rate and lower inflation than the bottom equilibrium (labelled

²²This conclusion follows from the existence of a $\underline{r}$ such that for all $r_t < \underline{r}$, households do not save, meaning that the household steady-state asset demand curve intersects the $b = 0$ axis at a finite interest rate $\underline{r}$. This discussion maintains the assumptions outlined in Section 3.1 so that $\mathbf{a}(r)$ is monotonically increasing.

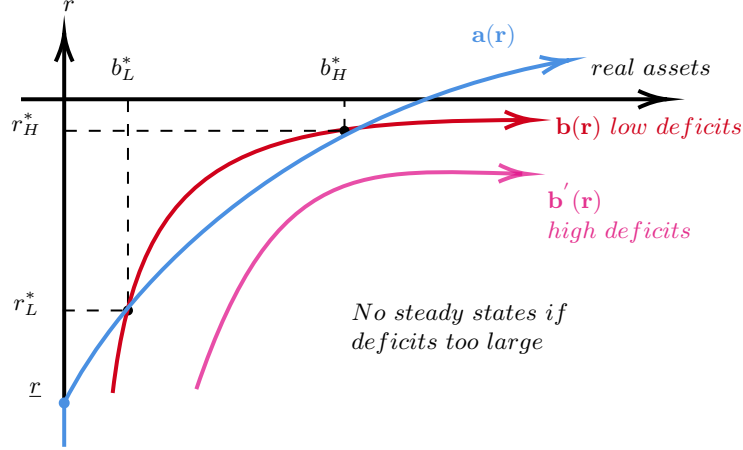


Figure 4: Maximum steady-state deficits

(b_L^*, r_L^*)). In Appendix A.6 we show that the high interest rate steady-state Pareto dominates the low interest rate steady-state. Intuitively, aggregate consumption is the same in both equilibria because this is a pure exchange economy. But with higher aggregate real assets, households are better able to insure against idiosyncratic shocks, so the variance of consumption growth is smaller and welfare is higher.²³

Maximum Deficits. The existence of multiple steady states is suggestive of the presence of a Laffer curve: the same amount of total spending can be achieved through a high quantity of real debt and a small negative rate of return or a lower quantity of debt and a larger negative return. As a corollary, there is a maximum level of deficits that is consistent with the existence of an equilibrium. As the level of deficits increases, the government asset supply curve shifts downward to the right, as illustrated in Figure 4. The maximum deficit is attained when the asset supply and demand curves are tangent to each other, which occurs at the point where the interest-rate elasticity of the steady-state household asset demand curve is equal to unity:

$$\frac{\mathbf{a}'(r)r}{\mathbf{a}(r)} = -1.$$

This condition reflects the fact that the maximum attainable level of deficits depends on the strength of households' desire to hold assets for precautionary reasons. It follows that a change in the nature of after-tax idiosyncratic endowment risk can shift the asset demand curve $\mathbf{a}(r)$ and hence the maximum deficit. For example, a reduction in s^* must be implemented via a change in the function $\tau^*(z)$. Depending on the change in progressivity, the maximum deficit may increase or decrease through a shift in $\mathbf{a}(r)$. In general, a change in the tax function that reduces the amount of uninsured risk will lower the maximum attainable deficit because households have less incentive

²³This argument relies on the zero borrowing limit. With a strictly negative borrowing limit, poor households may prefer lower interest rates to finance their debt.

to accumulate precautionary savings. In Section 5 we use our calibrated model to illustrate these effects.

Non-uniqueness of Price Level and Inflation. Since there are two steady-state equilibria with $s^* < 0$, it immediately follows that standard FTPL arguments for uniqueness of the price level do not hold. Additional assumptions on fiscal or monetary policy must be imposed, or other modifications made to the economy, in order to uniquely pin down the price level and inflation. We discuss these possibilities in Section 4.3, but first we characterize the set of non-stationary equilibria.

4.2 Non-stationary Equilibria

Local Dynamics. We can characterize the local dynamics around each of the two steady states using the same line of argument as we did for the case with $s^* > 0$. The dynamics obey the same PDE system (28). The arguments we gave for why the eigenvalues associated with the backward looking component $f(\omega, z)$ are all non-negative remain unchanged. As before, we sign the eigenvalue associated with the jump variable b_t by assuming that – in the vicinity of a steady-state equilibrium – the effect on government debt dynamics due to general equilibrium feedback from movements in the distribution are small relative to the overall effect of changes in interest payments:

$$\frac{db_t}{dt} \approx b^* \left\{ \partial_b \mathbf{r} [p^*, b^*] - \left(\frac{r^*}{b^*} \right) \right\} \quad (34)$$

The term in braces is the difference between the slopes of the steady-state asset demand function ($\partial_b \mathbf{r}[\Omega^*] = (\partial_r \mathbf{a}[r^*])^{-1}$) and the steady-state bond supply function ($-\frac{r^*}{b^*} = (\partial_r \mathbf{b}[r^*])^{-1}$). The eigenvalue associated with government debt b_t is therefore positive at the top steady-state, where the asset demand function crosses the asset supply function from below, and is negative at the bottom steady-state, where where the asset demand function crosses the asset supply function from above. Hence the local dynamics around the top steady-state are saddle-path stable, similarly to the unique steady-state in the case with surpluses. The dynamics around the bottom steady-state are locally stable. Simulations confirm these properties.

Figure 5 illustrates these dynamics. For a given initial distribution $f_0(\omega, z) \neq f^*(\omega, z)$, there is a unique equilibrium converging to (b_H^*, r_H^*) and a continuum of equilibria converging to (b_L^*, r_L^*) , indexed by the initial level of real debt b_0 . Each of these equilibria has different paths of associated real interest rates. Therefore the price level and inflation are not pinned down without additional assumptions that rule out almost all of these equilibria. Because it is the top equilibrium that is saddle-path stable, there is a lower bound on the initial price level that is consistent with equilibrium. This minimum initial price level is given by $P_0 = \frac{B_0}{b_0}$, where b_0 is the unique initial value of real debt for which the economy converges to the top equilibrium.²⁴

²⁴The existence of this lower bound descends from the fact that any equilibrium path that converges to the high-inflation steady-state must have an initial value of real debt lower (i.e., to the left) of the one corresponding to the saddle-path stable equilibrium because otherwise it would cross the saddle path.

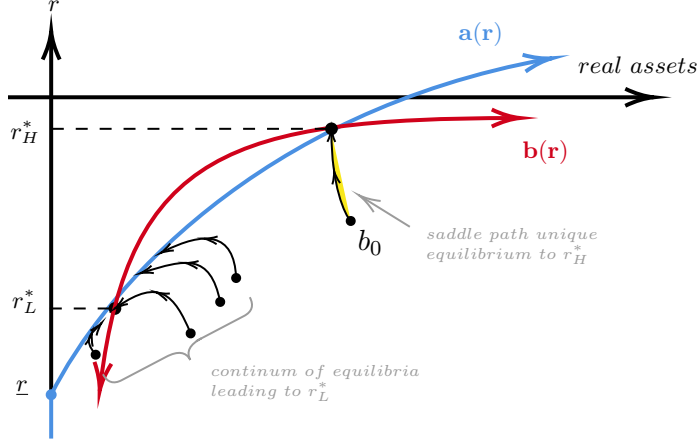


Figure 5: Non-stationary equilibria with deficits. For a given $f_0(\omega, z) \neq f^*(\omega, z)$, there are a continuum of equilibria indexed by the initial value of real government debt.

Exact Characterization in Representative Agent Bonds-In-Utility Economy. In Appendix D.4 we show that the representative agent economy with bonds in the utility function has qualitative steady-state properties that are the same as in the heterogeneous agent economy. In that economy, $\mathbf{a}^{BIU}[r]$ can be derived in closed form, and we can fully characterize the global dynamics. Analogous properties hold: the top steady-state is unstable, the bottom steady-state is stable and there is a lower bound on the initial price level

4.3 Options for Price Level Determination

Multiplicity of equilibria poses a challenge for quantitative work with this model. However, there are several ways to deliver uniqueness. We first consider two classes of fiscal policy rules that can eliminate the locally stable steady-state. We then consider eliminating the locally stable steady-state by introducing a foreign sector with relatively inelastic demand for domestic government debt. Lastly, we propose a form of long-run inflation anchoring that eliminates the equilibria leading to the low steady-state by coordinating long-run expectations about inflation.

Real Debt Reaction Rule. Until now we have assumed a fiscal rule that keeps primary deficits constant. Assume instead that the fiscal authority follows a rule in which primary deficits s_t respond to deviations of real debt from its steady-state level b^* ,

$$s_t = s^* + \phi_b (b_t - b^*). \quad (35)$$

The steady-state level of deficits is denoted by $s^* < 0$. Outside of steady-state, the fiscal authority varies deficits by appropriate changes in the tax and transfer function $\tau_t(z)$. The steady-state government asset supply curve is given by $r = \phi_b + \frac{s^* - \phi_b b^*}{b}$. If $\phi_b < r^* < 0$, then for $b < 0$, this is a downward sloping curve that intersects the household asset demand curve only once, as illustrated

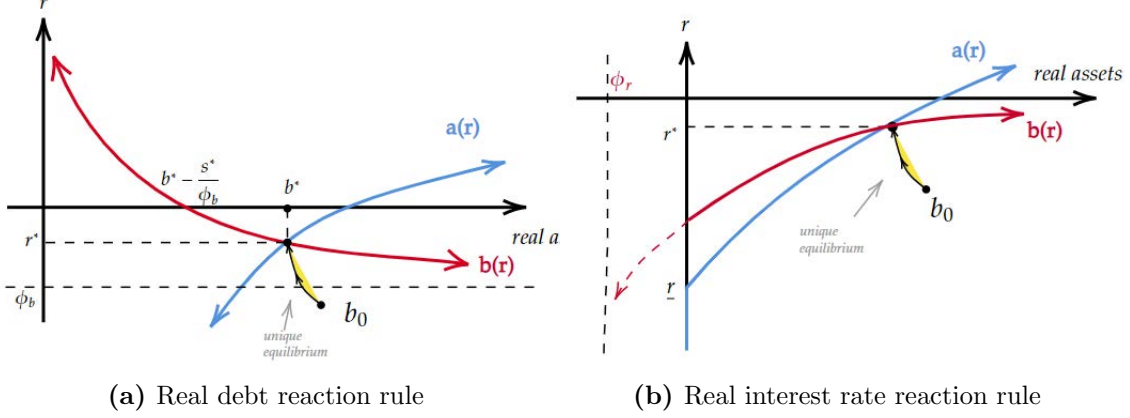


Figure 6: Equilibrium dynamics with deficits and fiscal rules

in Figure 6a.²⁵ There is a unique steady-state equilibrium which is saddle-path stable and hence the initial price level and subsequent inflation are uniquely determined. Appendix A.7 contains details.

The condition $\phi_b < r^*$ implies that when outstanding debt falls below its steady-state level, the government commits to respond by cutting primary deficits. This has a destabilizing effect on the debt accumulation process, which eliminates the bottom (stable) steady-state (b_L^*, r_H^*) .

This rule has the somewhat unappealing feature that when government debt rises above its steady-state level, the government responds by running even larger primary deficits. However, this property is not important for uniqueness; the role of the rule is to eliminate the stable equilibrium with *low* levels of government debt. Upward explosive dynamics are ruled out even with a constant deficit policy as explained in Section 3. For example an asymmetric policy, in which primary deficits respond only to reductions in government debt would suffice for uniqueness.

Real Rate Reaction Rule. An alternative fiscal rule that also eliminates the stable steady-state equilibria is one in which primary deficits respond to deviations of the equilibrium real rate from its steady state,

$$s_t = s^* + \phi_r (r_t - r^*). \quad (36)$$

In Appendix A.8, we show that a sufficient condition to eliminate the stable steady-state is $\phi_r < \frac{s^*}{r^* - a^{-1}(0)} < 0$. The resulting steady-state asset supply and demand curves are displayed in Figure 6b.

When the equilibrium real rate falls below its steady-state value, the fiscal authority responds by cutting its primary deficits. This has the effect of speeding up the corresponding fall in government debt which has a destabilizing effect that eliminates the bottom stable steady-state.

²⁵The household asset demand curve will also be affected, since higher levels of debt are associated with different transfer functions, which may alter the shape of the asset demand curve. In practice this effect can be made small by changing the level of deficits in an approximately distributional neutral way.

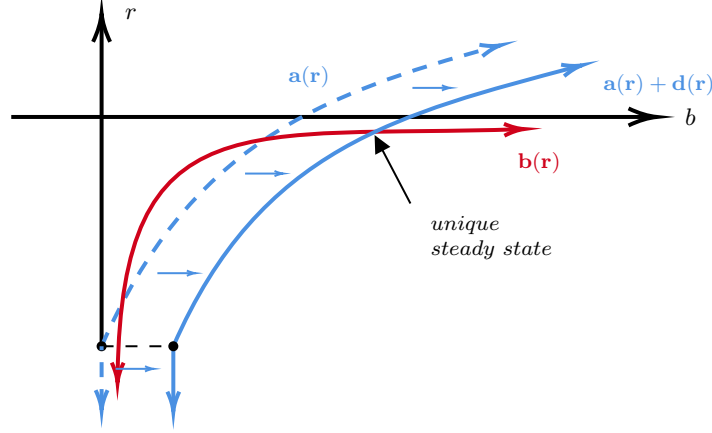


Figure 7: Inelastic foreign demand for government bonds

Interest Payment Reaction Rule. Our baseline analysis with a constant surplus s^* is a special case of a rule in which primary surpluses respond to deviations of real interest payments from their steady state level, as in equation (33) when $\phi_s = 0$. In Appendix A.9 we show that for all values of $\phi_s \neq 1$, the steady-state equilibria are unchanged from this baseline. As in the case with surpluses, with an active rule ($\phi_s < 1$), the stability properties of the two steady-states are also unchanged. However, with a passive fiscal rule ($\phi_s > 1$), the stability properties of the two steady-states are reversed: the top steady-state is locally stable and the bottom steady-state is saddle-path stable. In this case, the initial price level is bounded from above by $P_0 \leq \frac{B_0}{B_L^*}$, rather than from below.

Hence with a constant deficit fiscal policy or an interest payment reaction rule, either active or passive, additional assumptions are required for uniqueness. We suggest two possibilities: (i) inelastic foreign demand, and (ii) long-run inflation anchoring.

Inelastic Foreign Demand. So far we have assumed that all demand for government debt comes from domestic households, who hold the debt for intertemporal and precautionary reasons. If there is additional demand for government debt that is sufficiently interest-inelastic, for example from a foreign sector, then the bottom steady-state can be eliminated and uniqueness restored.

Denote the foreign demand for government debt as a function of the domestic real interest rate as $\mathbf{d}(r)$. The asset market clearing condition becomes

$$\mathbf{a}(r) + \mathbf{d}(r) = \mathbf{b}(r)$$

To see clearly the effect of additional foreign demand, assume that it is perfectly inelastic, so that $\mathbf{d}(r) = b^f$. The overall asset demand curve is shifted to the right and the bottom steady-state disappears, as illustrated in Figure 7. The assumption of a perfectly inelastic demand for foreign debt is extreme. In Appendix B.4 we offer a microfoundation based on a representative agent foreign sector that has bonds-in-utility preferences. We show that as long as there is positive demand for debt at all interest rates, and that the interest elasticity of demand is below one, then

the two curves intersect only once and there is a unique equilibrium. However, if there is a finite real interest rate below which even the foreign sector refuses to hold government debt, then the bottom steady state survives and the price level is not uniquely determined.

Long-Run Inflation Anchoring. The previous approaches to delivering a unique path of prices work by making assumptions that eliminate the high inflation stable steady-state, leaving only the low inflation saddle-path stable steady state. An alternative route to uniqueness is to instead eliminate all dynamic equilibria that lead to the high inflation steady-state, leaving only the unique equilibrium leading to the low inflation steady-state.

One way to achieve this is to endow the monetary authority with the power to coordinate private sector beliefs about long-run inflation. Under such a setting we envisage two pillars of central bank policy: (i) a path or rule for short-term nominal interest rates i_t , and (ii) a long-run inflation target π^* . Whereas the interest rate is a policy tool that the central bank directly implements by intervening in appropriate markets or paying interest on reserves, the long-run inflation target is no more than an attempt to coordinate beliefs. If:

- (i) the long-run inflation target and the long-run nominal interest rate (π^*, i^*) are set to be consistent with the equilibrium real rate at the saddle-path steady state, $i^* - \pi^* - g = r_H^*$;
- (ii) fiscal policy follows a constant deficit policy or a passive interest payment reaction rule with $\phi_s < 1$, so that the high real rate, low inflation steady-state is saddle-path stable;
- (iii) private sector beliefs about long-run inflation are consistent with the central bank's target,

then there is a unique real equilibrium and the price-level and inflation are pinned down for all t . The third of these conditions is a big “if”, and there is no fundamental reason to expect it to hold. However the key point is that managing *long-run* inflation expectations is sufficient to pin down the price level and inflation in the *short-run*. If the central bank is successful at convincing the private sector to coordinate on a long-run inflation target, then this is sufficient to eliminate any indeterminacy about inflation at all points in time. Note that anchoring long-run inflation expectations at π^* does not assume away the issue of price-level determination in the short-run. Both the initial price level and subsequent inflation remain endogenous and depend on monetary policy, fiscal policy and private sector behavior.

Even with long-run inflation anchoring, fiscal policy remains an essential component of price-level determination. Coordinating long-run expectations only uniquely determines the price-level in the short-run if fiscal policy acts in a way that ensures the saddle-path stability of the low-inflation steady state. Such fiscal policy settings are the same as those required for uniqueness in the case with persistent surpluses.

5 Quantitative Exercises with Persistent Deficits

In this section we report results from quantitative experiments for a calibrated version of the model with persistent deficits. These simulations illustrate the role of redistribution and precautionary

saving in shaping price level dynamics.²⁶

5.1 Model Extensions

The quantitative version of the model extends the baseline framework in two directions.

Extension I: Unsecured Household Credit. We allow for a non-zero borrowing limit. This permits nominal positions to be negative, thereby allowing some households to experience a positive wealth effect from an unanticipated rise in the price level, as in [Doepke and Schneider \(2006\)](#) and [Auclert \(2019\)](#). We assume that households can borrow up to a fixed limit that is denominated in real terms. We interpret it as unsecured borrowing, such as credit card debt, and impose an exogenous wedge between borrowing and saving rates. See [Appendix B.2](#) for a full description of the model with credit.

Extension II: Long-Term Debt. We assume that the government issues long-term debt with a constant maturity rate. The switch from short-term to long-term debt has no effect on the preceding qualitative analysis about price level determination. However, as shown by [Sims \(2011\)](#) and [Cochrane \(2018\)](#), debt duration plays an important role in determining the dynamics of inflation after unanticipated changes in the nominal interest rate. This mechanism surfaces in some of our experiments where we explore monetary policy rules beyond an interest rate peg. [Appendix B.3](#) describes the model with long-term debt.

5.2 Parameterization

Preferences. We set the elasticity of inter-temporal substitution γ to 1 so that households have log utility. We choose the discount rate ρ to match an annual debt-to-GDP ratio of 1.10 in the low inflation steady state. This target, which corresponds to the debt-to-GDP ratio in US data for the years leading up to the pandemic (2014-2019), implies a calibrated annual discount rate of 2.8%.

Endowment Process. We assume an annual aggregate real growth rate g of 2%, which was the US per-capita average over the post-war period.²⁷ Idiosyncratic endowment shares follow an $N_z = 5$ state process, with switching rates chosen so that income shocks arrive on average once per year and the implied endowment process generates a standard deviation of log quarterly earnings of 1.08, in line with US micro data.²⁸

Household Borrowing. We set the borrowing limit $\underline{a}$ to \$15,000, which is approximately 70% of average quarterly household earnings to match the median credit card limit for working-age population in the Survey of Consumer Finances (SCF) ([Kaplan and Violante, 2014](#)). We set the wedge between the interest rates on borrowing and saving to 16% p.a., based on typical interest

²⁶Our economy is a flexible price, endowment economy in continuous time. In reality, the price level does not jump. Rather, the initial bursts of inflation from these shocks are drawn out over a period of time. Despite this simplification, the general forces at work are informative about the two-way feedback between the equilibrium wealth distribution and movements in the price level.

²⁷See Series A939RX0Q048SBEA.PC1 from FRED, Federal Reserve Bank of St. Louis, <https://fred.stlouisfed.org>.

²⁸See, for example, the Global Repository of Income Dynamics (GRID), <https://www.grid-database.org/>.

Parameter	Value	Target
Preferences		
γ Inverse EIS	1	
ρ Discount rate	2.8% p.a.	debt-to-annual GDP ratio of 1.10
Income Process		
g Real output growth	2.0% p.a.	average growth rate post-war
λ Arrival rate of earnings shocks	1.0 p.a.	
σ St. Dev. of log quarterly earnings	1.2	
Household Borrowing		
$\underline{a}$ Borrowing limit	\$15,000	70% of quarterly household earnings
$r^b - r$ Borrowing wedge	16% p.a.	average rate on credit card debt
Tax and Transfers: $\tau(z) = \tau_0 - \tau_1 * z$		
τ_1 Proportional tax rate	30%	personal taxes / labor income
τ_0 Lump sum transfer	33.3% of GDP	deficit: $s^* = -3.3\%$
Government Debt		
δ Maturity rate of government debt	20% p.a.	average duration of 5 years
Monetary Policy		
i Nominal rate	1.5%	average Federal Fund Rate

Table 1: Calibrated parameter values and targets.

rates on unsecured credit card debt.²⁹ Because of this exogenous wedge, the real borrowing rate is positive, and the natural borrowing limit is finite and exceeds the ad-hoc limit.

Tax and Transfer System. The tax and transfer system consists of a lump-sum transfer and proportional tax,

$$\tau(z) = -\tau_0 + \tau_1 z.$$

We set the proportional tax rate τ_1 to 30% to match the ratio of personal taxes and social insurance contributions to total labor income (NIPA Table 2.9) for 2014-2019. We then set the lump-sum transfer τ_0 at 33.3% of aggregate output to generate a primary deficit s^* of -3.3% of GDP, which was the average for the US over that same period.³⁰

Government Debt. We assume that 20% of outstanding government debt matures each year to match a weighted average duration of 5 years (US Treasury). Given our target debt-to-GDP ratio of 110%, and primary deficit of 3.3%, the implied steady-state real interest rate is $r = \frac{s^*}{B^*} + g = \frac{-0.033}{1.1} + 0.02 = -1\%$ p.a.

²⁹See Table Consumer Credit - G19, Federal Reserve Board, <https://www.federalreserve.gov/releases/g19/current/>.

³⁰The data sources for debt and deficits are series GFDEGDQ188S and FYFSGDA188S from FRED.

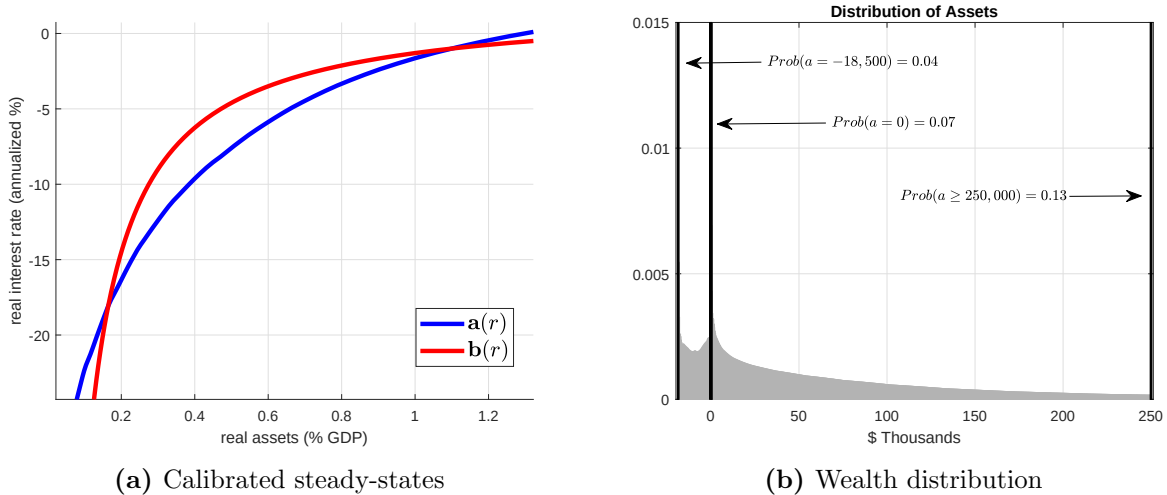


Figure 8: Calibrated steady-states and wealth distribution

Monetary Policy. We assume that the central bank pegs the nominal rate at 1.5% p.a., consistent with the average interest rate target in the years leading up to the pandemic. With a real interest rate of -1% , the implied steady-state annual inflation rate is 2.5%.

5.3 Properties of Steady States

Figure 8a displays the two stationary equilibria implied by our calibration. In line with our targets, the low inflation saddle-path stable steady-state has an annual debt-to-GDP ratio of 110% and an annual inflation rate of 2.5%. The high inflation locally stable steady-state has an annual debt-to-GDP ratio of 17.5%, and an annual inflation rate of around 19.5%. In what follows, we focus on the low-inflation steady state.

Wealth and MPC Distribution. Figure 8b and Table 2 illustrate that the calibrated wealth distribution is broadly consistent with the distribution of liquid wealth in the 2019 SCF.³¹ We exclude the top 1% of households in the SCF by liquid wealth because of the well-known difficulties in matching the right-tail of the wealth distribution in this class of models. Expressed in 2019 dollars, mean and median household wealth in the model are \$116,000 and \$40,000 respectively. 19% of households have negative wealth and 27% of households have less than \$1,000. These moments were not targeted in our calibration, which was disciplined by aggregate statistics on national debt.

The average quarterly MPC in the model is around 14%, with the highest MPCs among the low-income households that either have close to zero wealth and so are near a kink in their budget constraint, or have substantial negative wealth and so are close to the borrowing limit.³²

³¹Our definition of liquid wealth includes money market, checkings, savings, and call accounts, as well as directly held mutual funds, stocks and bonds, minus credit card and uncollateralized debt.

³²In Appendix A.10, we report additional details on the distributions of wealth and marginal propensities to consume in the model.

Table 2

Mean liquid assets	Data	Model
Mean assets	\$116,000	\$100,317
Frac. with $a < \$0$	20.67%	19%
Frac. with $a < \$1,000$	37%	27%

Note: Moments of the wealth distribution in the model and the data. Monetary values expressed in 2019 dollars. Data is from the 2019 Survey of Consumer Finances (SCF) with the top 1% of households by liquid wealth are excluded. See the main text for the definition of liquid assets in the data.

Maximum Sustainable Deficit. As discussed in Section 4.1, there exists a maximum possible level of permanent deficits consistent with existence of an equilibrium. The size of this maximum deficit depends on whether it is generated by expanding lump-sum transfers or cutting proportional taxes. Under our calibration, raising transfers yields a maximum deficit of 4.6% of output, a 39% increase from the baseline steady-state value of 3.3%. Instead, lowering taxes allows the government to run a maximum deficit of 4.8%, a 45% increase from the baseline.

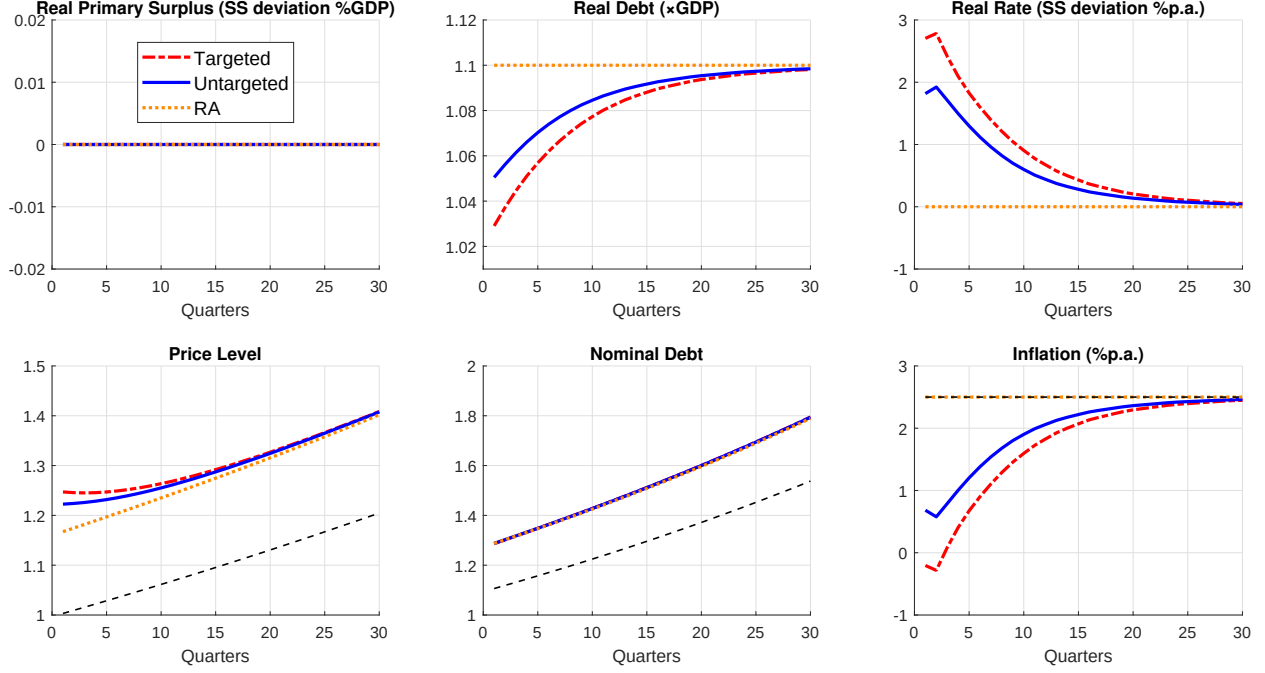
Lower proportional tax rates are, in general, associated with higher maximum steady-state deficits because they increase the volatility of disposable earnings. Households therefore bear more uninsured idiosyncratic income risk which raises their overall precautionary demand for safe liquid assets. For a given interest rate r , households are willing to hold more government bonds if they bear more idiosyncratic risk, giving the government more room to expand its deficit. Graphically, a lower value for τ_1 induces an outward shift in the the steady-state household asset demand curve (recall Figure 4). The same logic, with signs reversed, applies to an expansion of lump-sum transfers by reducing the variance of net earnings.

The role of precautionary saving is quantitatively important. For example, in an extreme case with only lump-sum transfers and no proportional taxes ($\tau_1 = 0\%$), the maximum sustainable deficit is 9.5%, which is almost three times as large as in our baseline. For similar reasons, when households are prohibited from borrowing, the maximum sustainable deficit rises to 5.9%.

A key lesson from these experiments is that reforms that loosen credit, make tax and transfer systems more progressive, or provide more insurance to households reduce future fiscal space available to the government. These reforms restrict the government’s ability to expand deficits or cut surpluses, and therefore may constrain the government’s ability to use expansionary fiscal policy to respond to adverse aggregate shocks.

Implications for Secular Stagnation. A recent literature argues that the secular decline of real rates observed in the US and other developed economies is due to rising income risk and inequality, which has been accelerated by the sharp debt deleveraging that occurred after the 2008 financial crisis (Auclert and Rognlie, 2018; Eggertsson et al., 2019; Mian et al., 2021b). The argument is that higher inequality leads to a redistribution of income from the high-MPC poor to the low-MPC rich, which increases overall demand for wealth in the household sector. Similarly, more uninsured income risk creates a stronger precautionary motive, which increases demand for

Figure 9



Note: This figure plots impulse responses to a targeted and untargeted helicopter drop, aggregated at the quarterly frequency. The helicopter drop is a one-time issuance of 16% of total government nominal debt outstanding at $t = 0$. Only households in the bottom 60% of the wealth distribution receive the issuance in the targeted experiment (dashed red line). See Footnote 33 for details regarding the representative agent calibration.

government bonds. A tighter borrowing limit induces households to save more for the same reasons. These forces all manifest as an outward shift of the household asset demand function $\mathbf{a}(r)$. In a conventional economy with positive rates and permanent surpluses, such outward shifts in household asset demand indeed leads to a lower steady-state real rate.

However, in an economy with permanent deficits and a negative real rate, all of these comparative statics are reversed when the economy starts in the low-inflation steady state. An outward shift of the household asset demand function $\mathbf{a}(r)$ leads to a higher steady-state real rate. The reason is that in order to finance the same level of deficits with a higher quantity of debt, a less negative (i.e. higher) real rate is needed. This observation adds an important qualification to the commonly held view that shifts in the income distribution, income risk or deleveraging are the primary explanation for secular stagnation. Below, in Section 5.5, we propose an alternative explanation for secular stagnation, rooted in the observation that in heterogeneous agent economies with persistent deficits and $r < g$, larger primary deficits result in a lower real interest rate.

5.4 Fiscal Helicopter Drop

Our first experiment is inspired by the experience of the US and other developed countries in the wake of the COVID-19 shock. In response to the disruptions caused by the pandemic, the US issued a large quantity of additional government debt and distributed much of the proceeds to households. We capture the core features of this fiscal helicopter drop by simulating an unexpected one-time issuance of nominal debt equal to 16% of initial outstanding government liabilities (equivalent to the observed 16% rise in the US debt-GDP ratio in 2020), which is distributed as a one-time lump-sum transfer to households. We consider two versions of this policy: one where transfers are distributed uniformly and one where transfers are distributed only to households in the bottom 60% of the wealth distribution, in line with the actual US experience.

Aggregate Effect of Fiscal Helicopter Drop. The effects of the fiscal helicopter drop are displayed in Figure 9. Since there are no changes to primary surpluses or any other structural parameters, the helicopter drop has no permanent real effects: the household and government nullclines are unchanged, and the economy converges back to its initial steady-state.

In the representative agent version of this economy, which is shown by the orange dotted line labelled “RA” in Figure 9, convergence is instantaneous.³³ The jump in the price level exactly offsets the new issuance of nominal debt so that the level of real debt remains constant and there are no further effects of the shocks.³⁴ However, in the heterogeneous-agent model there are transitional dynamics. The computed saddle-path dynamics associated with this convergence in (r_t, b_t) space are displayed in Figure 10. The initial jump in the price level is about 21%, more than in the representative agent model, which more than offsets the 16% rise in nominal debt.

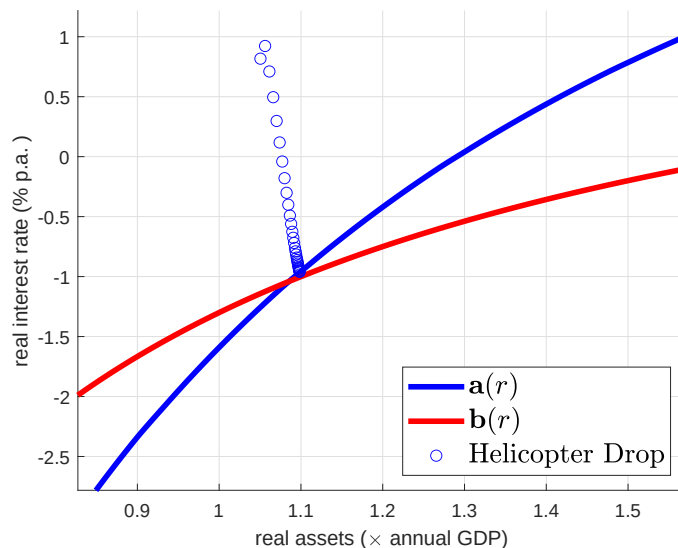
Why does an identical expansion in government debt place more upward pressure on the price level in the heterogeneous agent economy? As explained in Section 3.3, the fiscal helicopter drop entails a redistribution of real wealth from high- to low-wealth households because the lump-sum transfer is progressive. Since the average MPC is higher among low wealth households, this redistribution raises the economy-wide desire to consume. With a constant aggregate endowment, the real interest rate must rise to restore goods market clearing. The higher (i.e. less negative) real interest payments require a reduction in the total amount of real government debt outstanding. Since nominal debt is fixed after the helicopter drop, the price level must then increase further. An alternative interpretation is simply that the additional spending pressure from redistribution, beyond the aggregate wealth effect, places more upward pressure on nominal prices than in the representative agent economy where only the wealth effect is present.

Decomposition of Fiscal Helicopter Drop. In addition to the the direct redistributive impact of the fiscal helicopter drop, there are two additional indirect general equilibrium channels at play

³³The representative agent economy is constructed to have the same steady-state debt-to-GDP ratio as in the heterogeneous agent economy. However, since the representative agent economy does not admit a steady-state with persistent deficits, we assume an annual surplus-to-GDP ratio of 3.3% and an equilibrium real rate of 1%. We adjust the nominal interest rate so that the inflation rate is the same in the two economies.

³⁴The initial price jump in Figure 9 is slightly more than 16% because in this and other figures, we plot impulse response functions aggregated to a quarterly frequency.

Figure 10

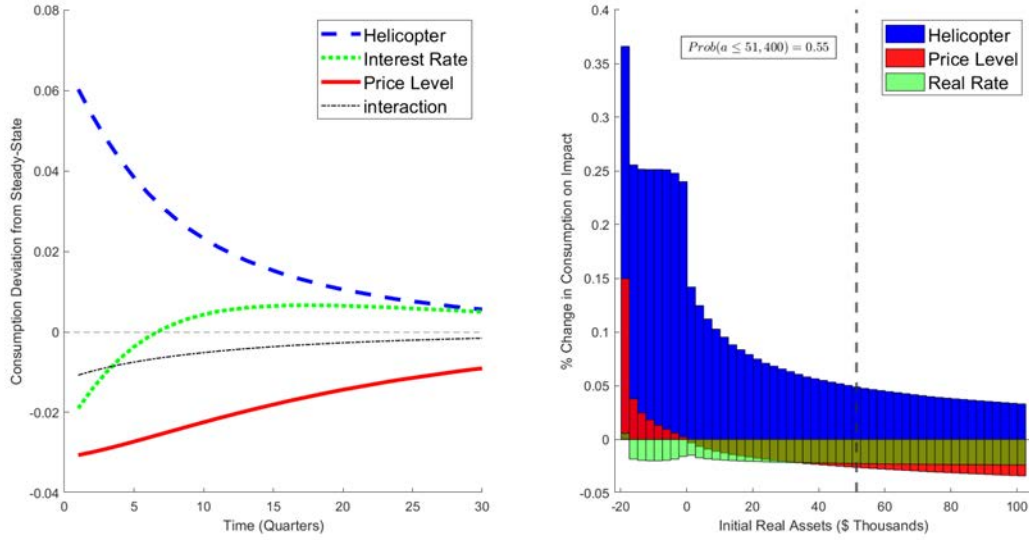


Note: This figure shows the computed saddle-path dynamics from a one-time issuance of nominal government debt in (r_t, b_t) space. The total issuance amounts to 16% of nominal government debt outstanding at $t = 0$. The blue dots depict quarterly aggregates.

that shape the subsequent dynamics of the real rate and inflation. First, the upward jump in the price level redistributes wealth from savers to borrowers, and dilutes the real savings for households with a positive net nominal position. Second, the resulting rise in the real rate leads households to postpone consumption. The left panel in Figure 11 displays the dynamic effects of each of these channels on aggregate consumption. The helicopter drop itself raises consumption, while the higher price level lowers consumption. These effects diminish as the economy returns to steady-state. The higher real interest rate leads households to delay consumption, which is reflected by the initially lower but subsequently higher consumption in the green dotted line in Figure 11.

The aggregate decomposition masks substantial heterogeneity in the effect of these channels across households. The right panel of Figure 11 shows the contribution of each channel to the change in consumption on impact along the wealth distribution. Low-wealth households increase consumption substantially, predominantly due to their higher MPCs out of the direct helicopter drop at the steady-state price level. In addition, the jump in the price level induces households with negative wealth to increase their consumption, because it lowers the real value of their debt. For households with positive wealth, the higher price level lowers their consumption because the real value of their nominal savings is reduced. The higher real interest rate lowers consumption for all households because of an intertemporal motive, except for households on the borrowing constraint. The dashed black line delineates the winners and losers of this experiment in terms of 2019 US dollars. Households with assets lower than \$51,400, which account for 55% of the population in our calibrated economy, gain from the helicopter drop.

Figure 11



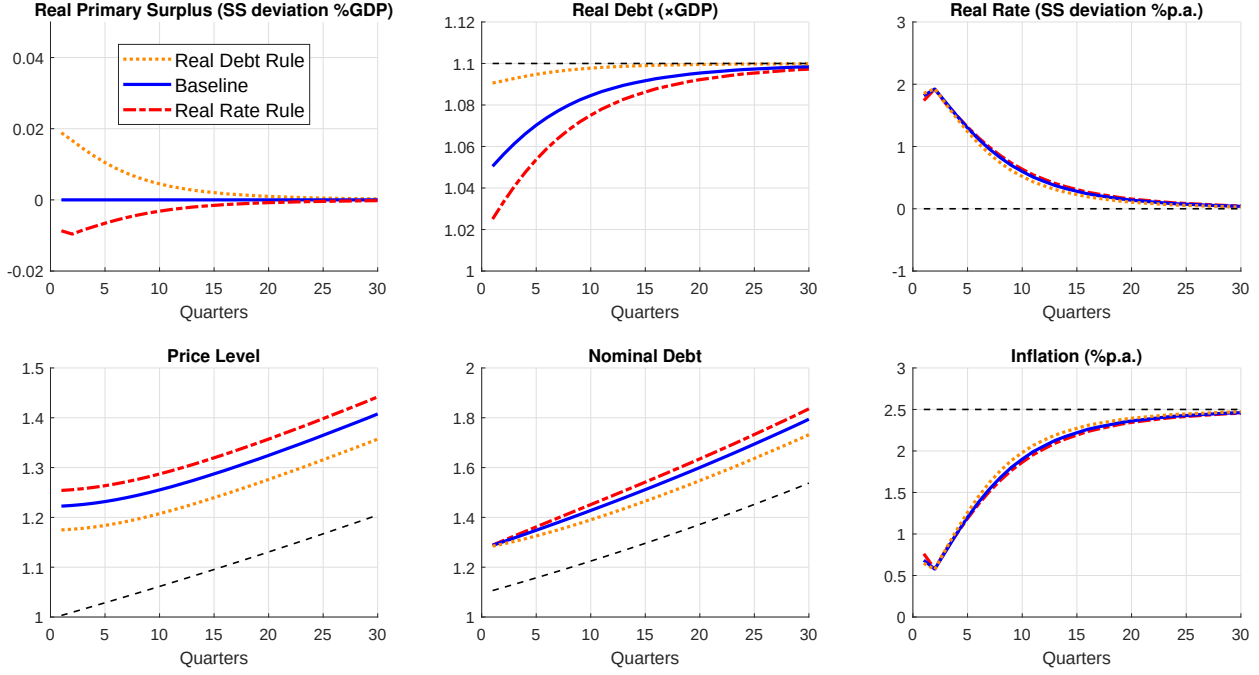
Note: This figure decomposes the effect of the helicopter drop on consumption into its general equilibrium sub-components. The left panel depicts how each sub-component affects aggregate consumption over time in isolation. The right panel depicts the effect of each sub-component on initial consumption across the wealth distribution. The dashed black line on the right panel delineates households that experienced initial consumption gains and losses as a result of the helicopter drop in 2019 US dollars.

Targeted vs Untargeted Fiscal Helicopter Drop. Figure 9 also shows that the initial increase in the price level is even larger when the helicopter drop is targeted towards poorer households. Compared to the untargeted case, the real interest rate increases by 1 additional percentage point on impact and, as a result, the price level jumps by an additional 4 percentage points (to 25%). In both the untargeted and targeted cases, the fiscal helicopter drop has a permanent effect on the price level and nominal government debt, but the inflationary effects are temporary. The saddle-path dynamics imply that both the real interest rate and the inflation rate return to their initial levels. In these experiments, the different price level responses between the heterogeneous agent and representative agent economies are mostly in terms of timing. The higher initial rise in prices in the heterogeneous agent economy is followed by lower inflation, so that the long-run cumulative increase in the price level is only slightly higher than in the representative agent economy.

Fiscal Helicopter Drop Under Different Surplus Reaction Rules. To justify focusing attention on the saddle-path equilibrium we are implicitly appealing to long-run inflation anchoring. As discussed in Section 4.3, surplus reaction rules are an alternative route to uniqueness. Figure 12 shows that the price level, real rate and inflation dynamics from the fiscal helicopter drop are not sensitive to using either of the two classes of surplus reaction rules in equations (35) and (36) that guarantee a unique equilibrium.

However, the two rules differ in the direction that primary deficits respond to the fiscal helicopter

Figure 12



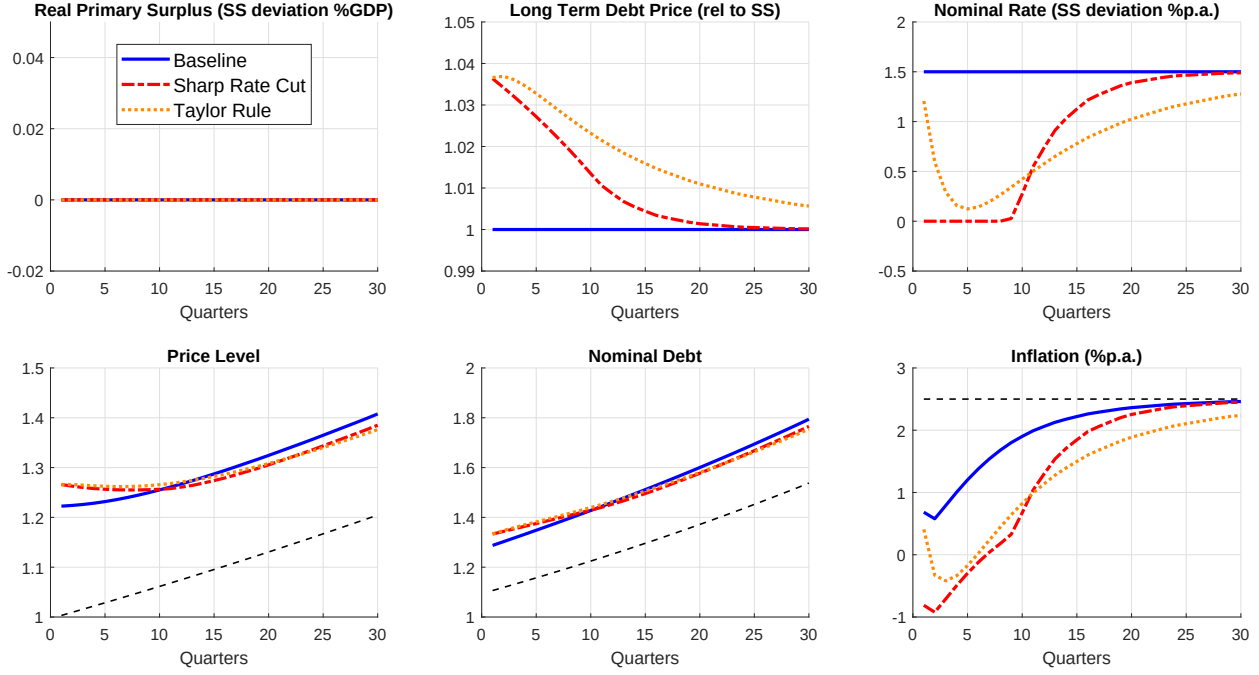
Note: Impulse responses to targeted fiscal helicopter drop under alternative fiscal rules. The dotted orange line corresponds to the “real debt rule” of equation equations (35) and the dashed red line corresponds to the “real rate rule” in equation (36) with parameter values of $\phi_b = -0.5$ and $\phi_r = -2$, respectively.

drop. Under the real debt reaction rule (35), the downward revaluation of real debt from the initial burst of inflation leads the fiscal authority to cut deficits following the helicopter drop. Under the real rate reaction rule, the higher real interest rate leads to a temporary increase in primary deficits.³⁵

Fiscal Helicopter Drop Under Different Monetary Responses. Throughout our previous simulations we have assumed that the central bank holds the nominal rate constant at 1.5% in response to the helicopter drop. Figure 13 reports impulse responses from two alternative experiments in which the monetary authority lowers interest rates at the same time as the fiscal expansion, like was done by central banks around the world in 2020. The dotted orange line labelled “Taylor rule” shows the effects of following a lagged Taylor rule as in equation (23), with a feedback parameter $\theta_m = 1$ and a coefficient on inflation $\phi_m = 0.5$. The dashed red line labelled “sharp rate cut” shows the implication of an even more powerful monetary accommodation of the fiscal expansion, corresponding to an immediate cut in the short-term interest rate all the way to zero, followed by a gradual normalization after 9 quarters. For comparison, the blue line labelled “baseline” reproduces

³⁵Cochrane (2023) argues that following an expansion in nominal debt, a reduction in primary deficits is more inline with the historical record for the United States. However, this is not always the case. See Jacobson et al. (2023) for an important historical example in which new debt was issued with the explicit intention of generating inflation by committing to not raise future surpluses to repay the debt.

Figure 13



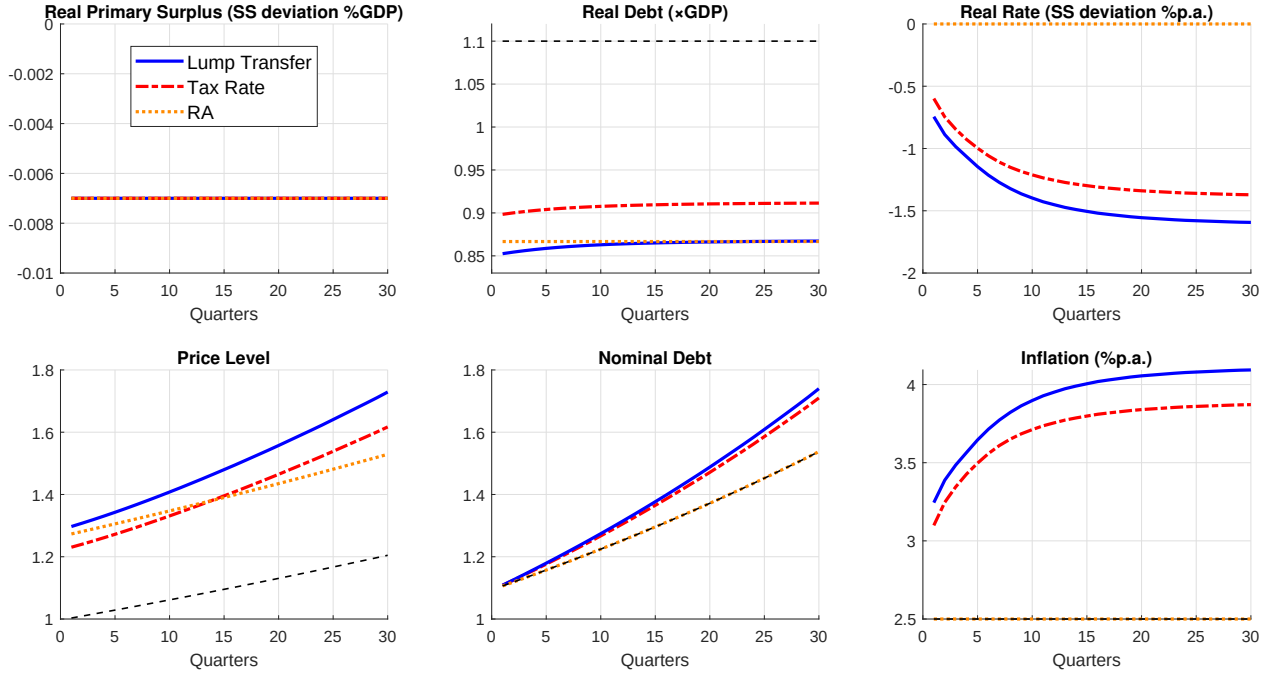
Note: Impulse response to targeted fiscal helicopter drop under different monetary policy responses. The dotted orange line corresponds to the Taylor rule in equation (23) with $\theta_m = 1$ and $\phi_m = 0.5$. The dashed red line is a temporary cut of nominal rates all the way to the zero lower bound.

the dynamics holding the nominal rate constant.

Monetary policy is a crucial driver of nominal aggregates. The behavior of long-term government bond prices is central to these dynamics. As explained in Sims (2011) and Cochrane (2018), a lower short-term nominal rate leads, through the yield curve, to a higher price of long-term government bonds. This means that the overall price level must rise by a larger amount to achieve the same-size drop in the real value of outstanding government debt. Figure 13 shows that looser monetary policy causes an additional 4 to 6 percentage point increase in the price level upon impact, relative to the baseline case with a nominal rate peg. The strength of this force is determined by the average duration of debt: the longer the duration, the bigger the initial jump in the price level. Different jumps in the price level, in turn, lead to different dynamics for real government debt and real interest rates through their effect on the real wealth distribution. However, we have found the effect on real variables to be quantitatively very similar across the three monetary specifications, provided that it is higher-wealth households that hold assets of longer duration.³⁶

³⁶If higher wealth households have longer duration portfolios, an unanticipated increase in monetary policy leads to larger capital losses for high-wealth households. However for moderate movements in the nominal rate, the relatively low MPCs of these households lead to small movements in the real rate. The assumption that high-wealth households hold relatively higher duration assets is consistent with empirical evidence. See Doepke and Schneider (2006).

Figure 14



Note: Impulse response to a permanent expansion in primary deficits. The dotted orange line shows the effects of a reduction in surplus in the Representative Agent model. The solid blue line labelled “Lump Sum” illustrates the dynamics following an expansion of lump sum transfers. The dashed red line labelled “Tax Rate” illustrates the dynamics following a tax cut.

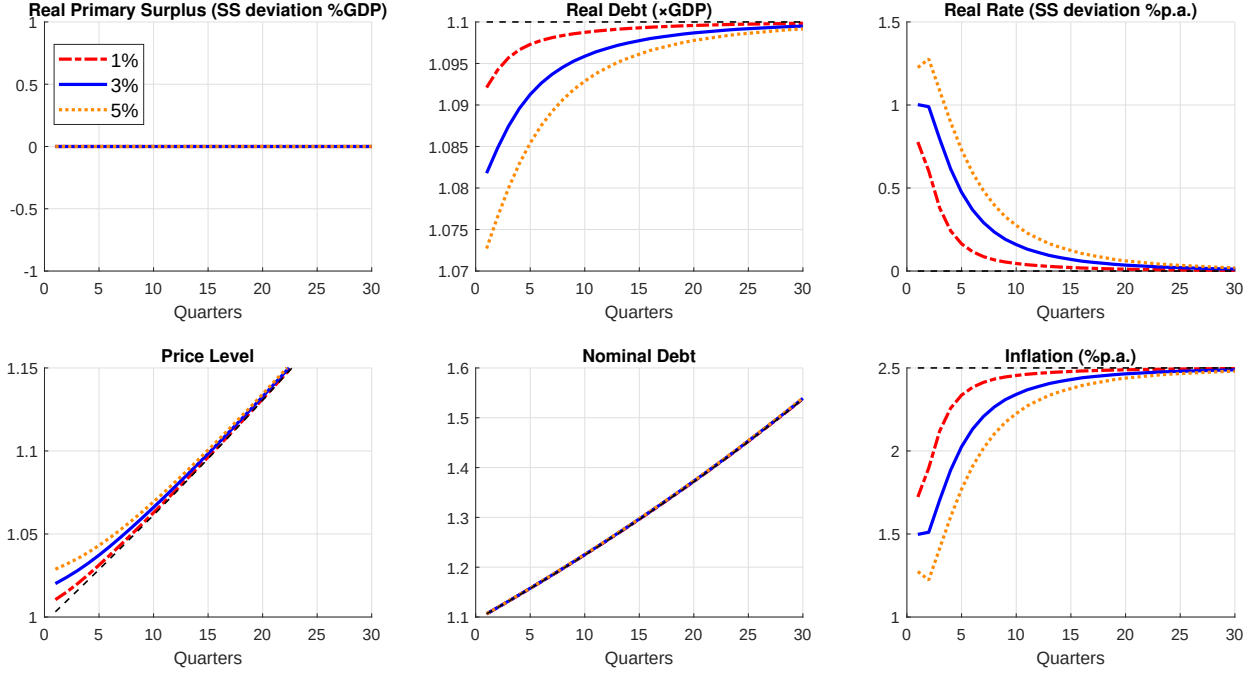
5.5 Permanent Deficit Expansion

Figure 14 displays impulse responses to a permanent deficit expansion from 3.3% to 4% of GDP. We consider two alternative policies for achieving a higher level of deficits. The solid blue line labeled “Lump-Sum” keeps the proportional tax rate the same and raises the lump-sum transfer. The dashed red line labeled “Tax Rate” reduces the proportional tax rate, while keeping lump-sum transfers at their initial level.

As was shown in Figure 4, a permanent increase in deficits shifts the steady-state government budget constraint downwards and to the right. Starting from the high real rate, low inflation steady-state, the long-run impact of the deficit expansion is to permanently lower both the real rate and the real value of government debt. These effects can be seen in the top row of Figure 14. The reduction in the value of real debt is achieved through a jump in the price level. In addition, because monetary policy does not respond, the lower real-rate translates into a permanently higher inflation rate. To prevent the permanent increase in deficits from leading to permanently higher inflation, the central bank would need to track the fall in the real rate by decreasing its target for the nominal rate.

Hence in the heterogeneous agent economy with deficits and negative real rates, a secular

Figure 15



Note: Impulse responses to a temporary increase in the wealth tax, with the proceedings distributed lump-sum, for various values of the wealth tax. In all experiments, the wealth tax is levied on the top 10% of the wealth distribution, the proceeds of which are redistributed lump-sum to the bottom 60%.

increase in primary deficits can account for a secular decline in real rates, i.e. secular stagnation. The fact that permanently higher deficits result in a permanently lower real rate and higher inflation is a distinguishing feature of the heterogeneous agent economy relative to the representative agent economy, in which a permanent increase in deficits has no impact on real rates or inflation.

These effects are all more pronounced when deficits are increased by raising lump-sum transfers than by lowering the proportional tax rate. The reason is that raising lump-sum transfers lowers the amount of uninsured idiosyncratic risk, thereby weakening the overall precautionary motive in the economy, while lowering proportional taxes raises the overall precautionary motive. Graphically, these differences manifest as different shifts in the household asset demand curve $\mathbf{a}(r)$.

5.6 Wealth Taxation

Inflationary Effects of Redistributive Wealth Taxes. A comparison of the heterogeneous agent and representative agent economies in the preceding experiments suggests that redistribution itself has effects on the price level and inflation that are independent of the overall level of surpluses and nominal government debt. To emphasize the inflationary effects of redistribution, Figure 15 shows simulations from purely redistributive shocks. We consider one-time wealth taxes levied on the top 10% of the wealth distribution, the proceeds of which are redistributed lump-sum to

the bottom 60%. Although these shocks do not entail any new issuance of government debt or any change in primary deficits, they do cause a period of inflation. The redistribution causes upward pressure on consumption because low-wealth households have higher average MPCs than high wealth households. Equilibrium is achieved through a period of higher real interest rates. The corresponding lower government revenues require a downward revaluation in real debt through a jump in the price level.

Inflationary Effects of Proportional Wealth Taxes. We contrast this experiment with another version of wealth taxation. Consider an economy where the government levies a proportional wealth tax at a rate of τ_b so that total primary surpluses are $s^* + \tau_b b_t$ (where s^* are surpluses net of revenue from the wealth tax). The real government budget constraint becomes:

$$db_t = [(r_t - \tau_b)b_t - s^*] dt. \quad (37)$$

The wealth tax appears in the household budget constraint in a similar fashion, as it increases the after-tax real rate paid to the government, $r_t - \tau_b$. Changes in τ_b therefore only affect the inflation rate through the Fisher equation, but otherwise leave the real economy and the initial price level unchanged.

5.7 Endogenous Output

We have so far focused only on endowment economies. In Appendix B.5 we describe an extension in which households make a labor-leisure choice and output is endogenous. This extension serves two purposes. First, it demonstrates that none of the qualitative forces relating heterogeneity and precautionary savings to prices and inflation that we have emphasized depend on an endowment economy per se. This is important because a natural next step is to extend our framework to one with nominal rigidities, in which endogenous output is essential.

Second, an important caveat to our analysis of permanent deficit expansions in Section 5.5 is that it abstracts from the effects of lower taxes and higher transfers on incentives to work. This rules out any changes in output that would lead to further endogenous changes in primary surpluses, thereby complicating the transmission from changes in the tax and transfer system to inflation. In Appendix B.5 we present simulations analogous to those in Figure 14, but in which labor supply responds endogenously. Our results suggest that output falls in the long-run following a deficit expansion, which amplifies the increase in deficits and leads to an even larger increase in the price level. Hours worked fall as a result of the permanent drop in the real interest rate which lowers the marginal return to work. When deficits are increased by raising lump sum transfers, there is an additional negative wealth effect on labor supply that further lowers output. When deficits are increased by cutting taxes instead, there is an offsetting effect on labor supply due to the higher incentives to work from a lower tax rate – but it is not strong enough to overcome the negative effect from a lower real rate.

6 Conclusions

We study the fiscal theory of the price level in a flexible price economy with heterogeneous agents and incomplete markets. In contrast to its representative agent counterpart, this model can be used to study an environment in which the government runs persistent deficits and the real rate is below the aggregate growth rate of the economy. This configuration is an accurate representation of the current state of affairs in many developed economies.

After showing that this model generically has two steady-states, we proposed a number of ways to obtain uniqueness for price level and inflation dynamics. Armed with uniqueness, we performed experiments that illustrate the forces at work in our model. The feature of our economy that accounts for different dynamics relative to its representative agent counterpart is the two-way feedback between price-level dynamics on the one hand, and redistribution and precautionary saving on the other. Redistribution and precautionary saving are also key determinants of the maximum deficit the economy can permanently sustain.

Future work should focus on formally obtaining sufficient conditions for saddle-path stability in both the surplus and deficit case. In on-going work we are extending this framework in two directions. The first is to include nominal rigidities, which gives rise to smoother price level dynamics. It also offers us the possibility to quantitatively confront the FTPL with the joint dynamics of inflation and output observed in the data, along the lines of what [Bianchi et al. \(2023\)](#) did in a representative agent model. The second is to extend our model to a two-asset economy with both low return nominal government bonds, and higher return real productive assets. Incorporating a two-asset household sector as in [Kaplan et al. \(2018\)](#) opens the door to a quantitative framework with a richer characterization of the possible assets through which households can save.

References

- Achdou, Y., Han, J., Lasry, J.-M., Lions, P.-L., and Moll, B. (2022). Income and wealth distribution in macroeconomics: A continuous-time approach. *The Review of Economic Studies*, 89(1):45–86.
- Aguiar, M. A., Amador, M., and Arellano, C. (2021). Micro risks and Pareto improving policies with low interest rates. National Bureau of Economic Research.
- Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684.
- Auclert, A. (2019). Monetary policy and the redistribution channel. *American Economic Review*, 109(6):2333–2367.
- Auclert, A. and Rognlie, M. (2018). Inequality and aggregate demand. National Bureau of Economic Research.
- Auclert, A., Rognlie, M., and Straub, L. (2018). The intertemporal Keynesian cross. National Bureau of Economic Research.
- Bassetto, M. and Cui, W. (2018). The fiscal theory of the price level in a world of low interest rates. *Journal of Economic Dynamics and Control*, 89:5–22.
- Benhabib, J., Bisin, A., and Zhu, S. (2015). The wealth distribution in Bewley economies with capital income risk. *Journal of Economic Theory*, 159:489–515.
- Bewley, T. (1983). A difficulty with the optimum quantity of money. *Econometrica*, 51(5):1485–1504.
- Bewley, T. (1987). Stationary monetary equilibrium with a continuum of independently fluctuating consumers. In Hildenbrand, W. and Mas-Colell, A., editors, *Contributions to Mathematical Economics in Honor of Gérard Debreu*. North-Holland Amsterdam.
- Bewley, T. (1995). The optimum quantity of money. In Kareken, J. and Wallace, N., editors, *Models of Monetary Economics*, volume 43, pages 1–46. USA: Federal Reserve Bank of Minneapolis.
- Bhatia, R. and Davis, C. (2000). A better bound on the variance. *The American Mathematical Monthly*, 107(4):353–357.
- Bianchi, F., Faccini, R., and Melosi, L. (2023). A fiscal theory of persistent inflation. *The Quarterly Journal of Economics*, page qjad027.
- Blanchard, O. (2019). Public debt and low interest rates. *American Economic Review*, 109(4):1197–1229.
- Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy*, 93(2):223–247.
- Brunnermeier, M. K., Merkel, S. A., and Sannikov, Y. (2020). The fiscal theory of price level with a bubble. National Bureau of Economic Research.
- Chetty, R., Guren, A., Manoli, D., and Weber, A. (2011). Are micro and macro labor supply elasticities consistent? A review of evidence on the intensive and extensive margins. *American Economic Review*, 101(3):471–475.
- Cochrane, J. (2021). r vs g . National Bureau of Economic Research.

- Cochrane, J. H. (1998). A frictionless view of US inflation. *NBER Macroeconomics Annual*, 13:323–384.
- Cochrane, J. H. (2001). Long-term debt and optimal policy in the fiscal theory of the price level. *Econometrica*, 69(1):69–116.
- Cochrane, J. H. (2018). Stepping on a rake: The fiscal theory of monetary policy. *European Economic Review*, 101:354–375.
- Cochrane, J. H. (2022). Inflation past, present and future: Fiscal shocks, Fed response, and fiscal limits. National Bureau of Economic Research.
- Cochrane, J. H. (2023). *The fiscal theory of the price level*. Princeton University Press.
- Coibion, O., Gorodnichenko, Y., Kueng, L., and Silvia, J. (2017). Innocent bystanders? Monetary policy and inequality. *Journal of Monetary Economics*, 88:70–89.
- Doepke, M. and Schneider, M. (2006). Inflation and the redistribution of nominal wealth. *Journal of Political Economy*, 114(6):1069–1097.
- Duffie, D. and Sun, Y. (2007). Existence of independent random matching. *Annals of Applied Probability*, 17(1):86–419.
- Duffie, D. and Sun, Y. (2012). The exact law of large numbers for independent random matching. *Journal of Economic Theory*, 147(3):1105–1139.
- Eggertsson, G. B., Mehrotra, N. R., and Robbins, J. A. (2019). A model of secular stagnation: Theory and quantitative evaluation. *American Economic Journal: Macroeconomics*, 11(1):1–48.
- Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5-6):953–969.
- Imrohoroglu, A. (1989). Cost of business cycles with indivisibilities and liquidity constraints. *Journal of Political economy*, 97(6):1364–1383.
- Jacobson, M. M., Leeper, E. M., and Preston, B. (2023). Recovery of 1933. National Bureau of Economic Research.
- Jappelli, T. and Pistaferri, L. (2010). The consumption response to income changes. *Annual Review of Economics*, 2(1):479–506.
- Kamihigashi, T. (2001). Necessity of transversality conditions for infinite horizon problems. *Econometrica*, 69(4):995–1012.
- Kamihigashi, T. (2003). Necessity of transversality conditions for stochastic problems. *Journal of Economic Theory*, 109(1):140–149.
- Kaplan, G., Moll, B., and Violante, G. L. (2018). Monetary policy according to HANK. *American Economic Review*, 108(3):697–743.
- Kaplan, G., Nikolakoudis, G., and Violante, G. (2023). The fiscal theory of the price level in heterogeneous agent New Keynesian economies. *In Progress*.
- Kaplan, G. and Violante, G. L. (2014). A model of the consumption response to fiscal stimulus payments. *Econometrica*, 82(4):1199–1239.
- Kaplan, G. and Violante, G. L. (2022). The marginal propensity to consume in heterogeneous agent models. *Annual Review of Economics*, 14:747–775.

- King, R. G., Plosser, C. I., and Rebelo, S. T. (1988). Production, growth and business cycles: I. The basic neoclassical model. *Journal of Monetary Economics*, 21(2-3):195–232.
- Kocherlakota, N. R. (2023). Public debt bubbles in heterogeneous agent models with tail risk. *International Economic Review*, 64(2):491–509.
- Leeper, E. M. (1991). Equilibria under ‘active’ and ‘passive’ monetary and fiscal policies. *Journal of Monetary Economics*, 27(1):129–147.
- Leeper, E. M. and Leith, C. (2016). Inflation through the lens of the fiscal theory. In Uhlig, H. and Taylor, J. B., editors, *Handbook of Macroeconomics*, volume 2. Elsevier Press: Amsterdam.
- Ljungqvist, L. and Sargent, T. J. (2018). *Recursive Macroeconomic Theory*. MIT press.
- McKay, A. and Wolf, C. K. (2023). Monetary policy and inequality. *Journal of Economic Perspectives*, 37(1):121–144.
- Mehrotra, N. R. and Sergeyev, D. (2021). Debt sustainability in a low interest rate world. *Journal of Monetary Economics*, 124:S1–S18.
- Mian, A., Straub, L., and Sufi, A. (2021a). A goldilocks theory of fiscal policy. National Bureau of Economic Research.
- Mian, A., Straub, L., and Sufi, A. (2021b). Indebted demand. *The Quarterly Journal of Economics*, 136(4):2243–2307.
- Miao, J. and Su, D. (2021). Fiscal and monetary policy interactions in a model with low interest rates.
- Pham, H. (2009). *Continuous-time stochastic control and optimization with financial applications*. Springer Science & Business Media.
- Reis, R. (2021). The constraint on public debt when $r < g$ but $g < m$. London School of Economics.
- Sargent, T. J. and Wallace, N. (1981). Some unpleasant monetarist arithmetic. *Federal Reserve Bank of Minneapolis Quarterly Review*, 5(3):1–17.
- Sims, C. A. (1994). A simple model for study of the determination of the price level and the interaction of monetary and fiscal policy. *Economic Theory*, 4:381–399.
- Sims, C. A. (2011). Stepping on a rake: The role of fiscal policy in the inflation of the 1970s. *European Economic Review*, 55(1):48–56.
- Stokey, N. L., Lucas, R. E., and Prescott, E. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.
- Woodford, M. (1995). Price-level determinacy without control of a monetary aggregate. In *Carnegie-Rochester Conference Series on Public Policy*, volume 43, pages 1–46. Elsevier.

Appendices

A Additional Model Details and Proofs

A.1 Derivation of Optimal Consumption Dynamics

This section derives expressions for the consumption dynamics of unconstrained constrained households. We also show that there exists an $\underline{r}$ such that households do not save for all $r \leq \underline{r}$.

Unconstrained Households. We show that the expected consumption dynamics for unconstrained households are given by

$$\frac{\mathbb{E}_t[dc_{jt}]}{c_{jt}} = \frac{1}{\gamma} (r_t - \rho) dt + \frac{1}{\gamma} \sum_{z'} \lambda_{z_{jt}z'} \left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}} \right)^{-\gamma} dt + \sum_{z'} \lambda_{z_{jt}z'} \left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}} \right) dt. \quad (38)$$

Here we use the short-hand notation $c_{jt} := c(a_{jt}, z_{jt}, \Omega_t)$ to denote the consumption of household j at time t .

Recall the HJB Equation:

$$\rho V_t(a, z) = \max_c u(c) + s_t(a, z) \partial_a V(a, z) + \sum_{z' \neq z} \lambda_{z, z'} [V_t(a, z') - V_t(a, z)] + \partial_t V_t(a, z) \quad (39)$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $s_t(a, z)$ is the savings function (11). The first-order condition is:

$$u'(c) = \partial_a V_t(a, z) \quad (40)$$

Differentiating the above with respect to a yields

$$u''(c_t(a, z)) \partial_a c_t(a, z) = \partial_{aa}^2 V_t(a, z) \quad (41)$$

Differentiating with respect to t yields

$$u''(c_t(a, z)) \partial_t c_t(a, z) = \partial_{at}^2 V_t(a, z) \quad (42)$$

The envelope condition for (39) is:

$$\rho \partial_a V_t(a, z) = \partial_{aa}^2 V_t(a, z) s_t(a, z) + r_t \partial_a V_t(a, z) + \sum_{z' \neq z} [\partial_a V_t(a, z') - \partial_a V_t(a, z)] + \partial_{at}^2 V_t(a, z) \quad (43)$$

Using (41) and (42) into the equation above yields:

$$\begin{aligned} (\rho - r_t) u'(c_t(a, z)) &= \sum_{z' \neq z} \lambda_{zz'} [u'(c_t(a, z')) - u'(c_t(a, z))] \\ &\quad + u''(c_t(a, z)) [\partial_t c_t(a, z) + s_t(a, z) \partial_a c_t(a, z)] \end{aligned} \quad (44)$$

This equation holds at any point on the interior of the state space $a > 0$ (i.e. for all unconstrained households). Using Ito's lemma for jump processes, it may equivalently be written as:

$$(\rho - r)u'(c_t(a_j, z_j)) = \frac{d\mathbb{E}[u'(c_t(a_j, z_j))]}{dt} \quad (45)$$

where we suppress the dependence of a_{jt} and z_{jt} on t for notational simplicity. Furthermore, using Ito's lemma on $c_t(a_j, z_j)$ yields

$$dc_j = \left[\partial_a c_t(a_j, z_j) s_t(a_j, z_j) + \partial_t c_t(a_j, z_j) + \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \right] dt + [c_t(a_j, z') - c_t(a_j, z_j)] d\tilde{N}_j \quad (46)$$

where $\tilde{N}_j$ is the compensated Poisson process for the stochastic process of income z' . Expected consumption therefore follows:

$$\mathbb{E}[dc_j] = \left[\partial_a c_t(a_j, z_j) s_t(a_j, z_j) + \partial_t c_t(a_j, z_j) + \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \right] dt \quad (47)$$

We may combine this with (44) to obtain

$$\begin{aligned} (\rho - r_t)u'(c_t(a_j, z_j)) &= \sum_{z' \neq z_j} \lambda_{z_j z'} [u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))] \\ &+ u''(c_t(a_j, z_j)) \frac{\mathbb{E}[dc_j]}{dt} - u''(c_t(a_j, z_j)) \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \end{aligned} \quad (48)$$

This yields (38) after dividing by $u''(c_t(a_j, z_j))$ and specializing to $u'(c) = \frac{c^{1-\gamma}}{1-\gamma}$

Constrained Households. We show that the expected consumption dynamics for borrowing constrained households satisfy:

$$\frac{\mathbb{E}_t[dc_{jt}]}{c_{jt}} = \sum_{z'} \lambda_{z_{jt}, z'} \left(\frac{c(a_{jt}, z', \Omega_t)}{c_{jt}} \right) dt. \quad (49)$$

The consumption dynamics for constrained households are given by

$$dc_t(0, z_j) = \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(0, z') - c_t(0, z_j)] dt + [c_t(0, z') - c_t(0, z_j)] d\tilde{N}_j \quad (50)$$

since households consume their income whenever constrained (until receiving a more favourable income draw). Taking expectations and dividing by $c_t(0, z_j)$ then yields (49) directly.

Existence of $\underline{r}$. This subsection shows that there exists a finite $\underline{r}$ such that no household saves in a stationary equilibrium if $r \leq \underline{r}$. Suppose no such $\underline{r}$ exists. Note that this implies that there

must exist a non-zero mass of households that are unconstrained in any stationary equilibrium, for all $r < \rho$.

Proposition 2 in [Achdou et al. \(2022\)](#) shows that there exists a finite upper bound on the state space for assets in a stationary equilibrium. Moreover, $\underline{z} > 0$. Hence, marginal utility and consumption are bounded from above and are strictly greater than zero for all a_{jt} and z_{jt} . Equation (38) then implies that there must exist an $\underline{r}$ such that

$$\frac{\mathbb{E}[dc_t(a_{jt}, z_{jt})]}{dt} < 0$$

for all households j that are unconstrained. But this would then imply that aggregate consumption must be decreasing, which would violate market clearing. Hence, there cannot exist a non-zero mass of households that are unconstrained in a stationary equilibrium with $r < \underline{r}$. But this implies the existence of such an $\underline{r}$, a contradiction.

A.2 Derivation of Real Rate Functional

This subsection derives the real interest rate functional given in Equation (26). We use (46) and (50) to integrate across all households j :

$$\begin{aligned} \frac{d}{dt} \int_j c_t(a_j, z_j) dj &= \int_{j:u} \left(\partial_a c_t(a_j, z_j) s_t(a_j, z_j) + \partial_t c_t(a_j, z_j) + \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(a_j, z') - c_t(a_j, z_j)] \right) dj \\ &\quad + \int_{j:c} \sum_{z' \neq z_j} \lambda_{z_j z'} [c_t(0, z') - c_t(0, z_j)] \end{aligned} \quad (51)$$

where the $d\tilde{N}_j$ terms vanish by the exact law of large numbers ([Duffie and Sun, 2007, 2012](#)). The first integral on the right-hand side is over unconstrained households ($j : u$), while the second integral is over constrained households ($j : c$). Note that the above equation must be equal to zero, since $\int_j c_t(a_j, z_j) dj = 1$, by market clearing. Dividing by $u''(c_t(a_j, z_j))$ in (44) and using CRRA preferences, we obtain:

$$\begin{aligned} -\frac{1}{\gamma}(\rho - r)c_t(a_j, z_j) &= \sum_{z' \neq z_j} \lambda_{z_j z'} \frac{1}{u''(c_t(a_j, z_j))} [u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))] \\ &\quad + \partial_t c_t(a_j, z_j) + s_t(a_j, z_j) \partial_a c_t(a_j, z_j) \end{aligned} \quad (52)$$

Integrating over all unconstrained agents and using (51) to substitute for $\partial_t c_t(a_j, z_j) + s_t(a_j, z_j) \partial_a c_t(a_j, z_j)$ yields

$$\begin{aligned} -\frac{1}{\gamma}(\rho - r) \int_{j:u} c_t(a_j, z_j) dj &= \int_{j:u} \sum_{z' \neq z_j} \lambda_{z_j z'} \frac{1}{u''(c_t(a_j, z_j))} [u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))] dj \\ &\quad - \int_j \sum_{z' \neq z_j} \lambda_{z_j z'} (c_t(a_j, z') - c_t(a_j, z_j)) dj \end{aligned} \quad (53)$$

Moreover, constrained agents consume their current income z_j . Hence, adding and subtracting $-\frac{1}{\gamma}(\rho - r) \int_{j:c} z_j dj$ to the equation above and rearranging yields an expression for the interest rate:

$$r = \rho + \gamma \frac{\int_{j:u} \sum_{z' \neq z_j} \lambda_{z_j z'} \frac{[u'(c_t(a_j, z')) - u'(c_t(a_j, z_j))]}{u''(c_t(a_j, z_j))} dj - \int_j \sum_{z' \neq z_j} \lambda_{z_j z'} (c_t(a_j, z') - c_t(a_j, z_j)) dj}{1 - \int_{j:c} z_j dj} \quad (54)$$

Using CRRA utility, and the fact that $\lambda_{z_j z_j} - \sum_{z' \neq z_j} \lambda_{z_j z'}$, we may write the above expression as

$$r = \rho - \frac{\int_{j:u} c(a_j, z_j) \left[\sum_{z'} \lambda_{z_j z'} \left(\frac{c(a_j, z')}{c(a_j, z_j)} \right)^{-\gamma} \right] dj + \gamma \int_j c(a_j, z_j) \left[\sum_{z'} \lambda_{z_j z'} \frac{c(a_j, z')}{c(a_j, z_j)} \right] dj}{1 - \int_{j:c} z_j dj} \quad (55)$$

Relative to the representative agent economy, the sum differs by two terms: the (i) marginal utility variation due to income risk for unconstrained agents, and (ii) consumption variation due to income risk for both constrained and unconstrained agents, multiplied by the coefficient of relative risk aversion. All of these terms are scaled by one minus the total income holdings of constrained agents (which is trivially less one since aggregate consumption is equal to one). The interest rate can be written as a functional in terms of aggregate states by replacing $c_t(a_{jt}, z_{jt})$ with $c(\omega_{jt} b_t, z_{jt}, \Omega_t)$. Equation (26) then follows directly.

A.3 Uniqueness with Constant Surpluses

In this section, we show that explosive paths for real assets are ruled out by the household transversality condition. Our proof strategy entails decomposing the expectation in (25) and aggregating across households to show that the rate of growth of aggregate assets is bounded below by the discount rate ρ .

Consider a strictly positive sequence of real rates $(r_t)_{t \geq 0}$. Recall that the transversality condition in the stochastic economy is:

$$\lim_{t \rightarrow \infty} [\mathbb{E}_t \exp(-\rho t) u'(c_t(a_{jt}, z_{jt})) a_{jt}] = 0 \quad (56)$$

The household Euler equation gives us a differential equation for the evolution of *expected* marginal

utility.

$$\frac{\mathbb{E}_0[du'(c_t(a_{jt}, z_{jt}))]}{u'(c_0(a_{j0}, z_{j0}))} = (\rho - r_t)dt \quad (57)$$

We may solve this ordinary differential equation to obtain

$$\mathbb{E}_0[u'(c_t(a_{jt}, z_{jt}))] = u'(c_0(a_{j0}, z_{j0})) \exp\left(\rho t - \int_0^t r_s ds\right) \quad (58)$$

We next decompose the expectation term in the household transversality condition:

$$\begin{aligned} \lim_{t \rightarrow \infty} [\mathbb{E}_0 \exp(-\rho t) u'(c_t(a_{jt}, z_{jt})) a_{jt}] &= \lim_{t \rightarrow \infty} [\exp(-\rho t) \mathbb{E}_0 [u'(c_t(a_{jt}, z_{jt}))] \mathbb{E}_0 [a_{jt}] \\ &\quad + \exp(-\rho t) \text{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt})] \end{aligned} \quad (59)$$

where the covariance is conditional on the households' time-zero information set. We may substitute for the first term using (58) to obtain:

$$\lim_{t \rightarrow \infty} \left[\exp\left(-\int_0^t r_s ds\right) u'(c_0(a_{j0}, z_{j0})) \mathbb{E}_0 [a_{jt}] + \exp(-\rho t) \text{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt}) \right] = 0 \quad (60)$$

We may also bound the covariance term via the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \exp(-\rho t) \text{Cov}_0(u'(c_t(a_{jt}, z_{jt})), a_{jt}) &\leq \exp(-\rho t) \sqrt{\text{Var}_0(u'(c_t(a_{jt}, z_{jt})))} \sqrt{\text{Var}_0(a_{jt})} \\ &\leq \exp(-\rho t) \frac{y_{min}^{-\gamma}}{2} \sqrt{\mathbb{E}_0[(a_{jt})^2]} \end{aligned}$$

where last the inequality has made use of the fact that $u'(c_{jt}) \leq y_{min}^{-\gamma}$ and the Popoviciu bound on variances (Bhatia and Davis, 2000). Finally, we provide a bound on the variance of individual asset holdings. If asset holdings are uniformly bounded, the bound is trivially zero. So we only need to concern ourselves with cases in which individual assets may diverge to infinity. In these cases, we can use standard results on the asymptotic behaviour of the consumption function to provide an upper bound on assets (Benhabib et al., 2015; Achdou et al., 2022). In particular, we have:

$$\lim_{a_{jt} \rightarrow \infty} \frac{\phi_t a_{jt}}{c_{jt}} = 1 \quad (61)$$

where $\phi_t > 0$. We may then use the household budget constraint to show that assets grow at a rate $r_t - \phi_t$ asymptotically, which yields the bound

$$a_{jt} \leq \Xi \exp\left(\int_0^t (r_s - \phi_s) ds\right), \quad \text{a.s.} \quad (62)$$

for some finite $\Xi > 0$. Using the Popoviciu inequality once again, we obtain

$$|\exp(-\rho t) \text{Cov}_0(u'(c_{jt}), a_{jt})| \leq \exp(-\rho t) \frac{y_{min}^{-\gamma}}{4} \Xi \exp\left(\int_0^t (r_s - \phi_s) ds\right) \quad (63)$$

Under the assumption that there exists some $t' > 0$ such that $r_t \leq \rho$ for $t \geq t'$, the right-hand side vanishes as we take $t \rightarrow \infty$. Section B.1 provides sufficient condition for $r_t < \rho$ for all $t \geq 0$.

We now show that (60) precludes explosive paths for real aggregate debt. In particular, we show that

$$\lim_{t \rightarrow \infty} \left[\exp \left(\int_0^t -r_s ds \right) a_t \right] = 0 \quad (64)$$

where a_t is the amount of aggregate asset holdings in the economy at time t . To this end, we integrate (60) over households to obtain:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left[\int_{a,z} \mathbb{E}_0 \exp(-\rho t) u'(c_0(a, z)) a dG_t(a, y) \right] \\ & \leq \lim_{t \rightarrow \infty} m \left[\exp \left(- \int_0^t r_s ds \right) \underbrace{\int_{a,y} \mathbb{E}_{a_0=a} [a_t] dG_t(a, z)}_{\bar{a}_t} \right] \\ & = 0 \end{aligned}$$

where m is an upper bound on marginal utility at $t = 0$: $u'(c_0^i(a^i)) \leq m \quad \forall a, y \in \text{supp } G_0(a, y)$, a.e. and where $G_t(\cdot, \cdot)$ is the distribution over assets and income at time t . Note that term in the integral in the second inequality is equal to aggregate asset holdings by the exact law of large numbers (Duffie and Sun, 2012). This shows that no equilibria exist in which government debt explodes upwards. Downward explosion paths are ruled out by the non-negativity constraint on aggregate real debt.

A.4 Finite Difference Approximation to Equilibrium Dynamics

We begin by deriving the Kolmogorov Forward Equation (KFE) for wealth shares. Note that the dynamics for wealth shares $\omega_{jt} = \frac{a_{jt}}{a_t}$ is given by

$$\frac{d\omega_{jt}}{\omega_{jt} dt} = \frac{da_{jt}}{a_{jt} dt} - \frac{db}{b_t dt} \quad (65)$$

Using Equations (6) and (18) yields

$$\frac{d\omega_{jt}}{dt} = \omega_{jt} \left(\frac{r_t a_{jt} + z_{jt} - \tau_t(z_{jt}) - c_{jt}}{a_{jt}} - \frac{r_t b_t - s_t}{b_t} \right) \quad (66)$$

$$\frac{d\omega_{jt}}{dt} = \frac{z_{jt} - \tau_t(z_{jt}) - c_{jt} + \omega_{jt} s_t}{b_t} \quad (67)$$

This implies that the KFE for wealth shares is given by:

$$\partial_t f(\omega, z) = \mathcal{A}_\omega^*[f, b](\omega, z) + \mathcal{A}_z^*[f](z) \quad (68)$$

where

$$\mathcal{A}_\omega^*[f, b](\omega, z) = \partial_\omega \left[f(\omega, z) \frac{z - \tau_t(z) - c_t(\omega, z; f, b) + \omega s_t}{b} \right] \quad (69)$$

and

$$\mathcal{A}_z^*[f](z) = -f(\omega, z) \sum_{z' \neq z} \lambda_{zz'} + \sum_{z' \neq z} \lambda_{z'z} f(\omega, z') \quad (70)$$

where we have made the dependence of the consumption function on aggregate state variables explicit. Note further that these operators are adjoint to underlying operators $\mathcal{A}_\omega$ and $\mathcal{A}_z$.

We may discretize the distribution $f(\omega, z)$ into $N = N_\omega \times N_z$ discrete points, where N_ω is a discrete grid for ω of width Δ_ω . We denote the discretized distribution as $\mathbf{f}$ and write the dynamics of the joint system as

$$\frac{d\mathbf{f}}{dt} = \mathbf{A}_\omega [\mathbf{f}_t, b_t]^T \mathbf{f}_t + \mathbf{A}_z^T \mathbf{f}_t \quad (71)$$

$$\frac{db}{dt} = \mathbf{r}[\mathbf{f}_t, b_t] b_t - s^* \quad (72)$$

The interest rate functional $\mathbf{r}[\mathbf{f}_t, b_t]$ corresponds to the interest rate functional in Equation (26) where we have substituted for the discretized endowment share distribution. The matrix $\mathbf{A}_\omega [\mathbf{f}_t, b_t]$ is a finite difference approximation to $\mathcal{A}[f, b]$ using the appropriate upwind scheme (Achdou et al., 2022). Hence, it is a tridiagonal matrix which consists of the following terms:

$$\left\{ 0, -\frac{z - \tau_t(z) - c_t(\omega, z; \mathbf{f}, b) + \omega s_t}{b \Delta_\omega}, \frac{z - \tau_t(z) - c_t(\omega, z; \mathbf{f}, b) + \omega s_t}{b \Delta_\omega} \right\} \quad (73)$$

The matrix $\mathbf{A}_z$ is the Markov transition matrix for z in the product space $\omega \times z$. Note that it is not indexed by z because the operator $\mathbf{A}_z$ is linear. The rows of both $\mathbf{A}_\omega [\mathbf{f}_t, b_t]$ and $\mathbf{A}_z$ sum to zero to ensure that $\mathbf{f}_t$ preserves mass.

The linearized system can be exactly expressed as (32) if the effect of $\mathbf{f}$ on the interest rate is small. A sufficient condition is that the real interest rate is invariant to changes in the endowment share distribution, which would occur if consumption functions were linear in wealth. However, because the interest rate functional uses a consumption-based aggregator, in practice it is only necessary for the consumption function to be linear amongst high-wealth households, who consume relatively more of the aggregate endowment.

A.5 Uniqueness with Zero Surpluses

The government accumulation equation with zero surpluses is

$$db_t = [\mathbf{r}(\Omega_t) b_t] dt \quad (74)$$

This implies a steady-state interest rate of $r^* = 0$ whenever $\mathbf{a}(0) > 0$, with an associated steady-state level of real debt given by $b^* \equiv \mathbf{a}(0)$. The first-order dynamics of this system around the

steady-state are given by:

$$db_t = [b^* \partial_b \mathbf{r}(\Omega^*)] dt \quad (75)$$

The last term is strictly positive due to household behaviour. Hence, the steady-state is locally saddle-path stable. Since $B_0 > 0$ is given, there exists a unique, finite value of P_0 such that the equilibrium converges back to the steady-state. There are also a continuum of stationary real equilibria with $P = \infty$, in which $r < \underline{r}$ and aggregate real debt is zero. This proves local uniqueness of the equilibrium. Conditions for global uniqueness are outlined in [Appendix A.3](#).

A.6 Steady-State Welfare Comparison

We show that steady-states with higher real interest rates are Pareto ranked for any initial condition of assets a_{jt} and income z_{jt} . In particular, consider a particular profile of income shocks $\{z_{jt}\}_{t \geq 0}$ that induces a (realized) consumption and savings streams $\{c_{jt}, a_{jt}\}$ under a constant real interest rate r_L^* . This consumption plan can also be implemented at a higher interest rate $r_H^* > r_L^*$ for the same sequence of income shocks, since the change in savings in any given period will be:

$$da_{jt} = [(r_H^* - r_L^*) a_{jt}] dt \quad (76)$$

which is weakly positive for any given $a_{jt} > 0$ (recall that the surplus s^* , and hence taxes and transfers, are fixed and independent of the level of the real interest rate). Higher interest rates weakly expand the budget set of all households for any given a_{j0} and z_{j0} . This proves that a steady-state with r_H^* Pareto dominates r_L^* .³⁷

A.7 Unique Steady State with Real Debt Reaction Rule

Our argument for uniqueness proceeds in three steps. First, we derive conditions for a unique steady-state. Second, we derive conditions for the steady-state to be saddle-path stable. This ensures local uniqueness. Finally, we consider whether explosive paths in debt can be ruled out globally. This ensures global uniqueness.

Steady-State Uniqueness. Suppose the government follows a fiscal rule of the form:

$$s_t = s^* + \phi_b(b_t - b^*) \quad (77)$$

where $s^* = r^* b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$db_t = [r_t b_t - s_t] dt \quad (78)$$

³⁷This proof strategy follows [Aguar et al. \(2021\)](#), who construct robust Pareto-improving policies in the presence of capital accumulation.

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{s^* - \phi_b b^*}{b} + \phi_b \quad (79)$$

A sufficient condition for steady-state uniqueness is that this function is downwards sloping. This will ensure that it intersects the upwards sloping steady-state demand curve $\mathbf{a}(r)$ exactly once. The slope of this function is

$$\frac{dr}{db} = -\frac{s^* - \phi_b b^*}{b^2} \quad (80)$$

$$= -\frac{r^* b^* - \phi_b b^*}{b^2} \quad (81)$$

which is strictly negative whenever $r^* > \phi_b$. Hence, $\phi_b < r^* < 0$ is sufficient for steady-state uniqueness.

Local Uniqueness. We now examine conditions for this fiscal rule to give rise to local uniqueness. Under our maintained assumptions on the dynamical system that obtain (32), local uniqueness amounts to checking whether the eigenvalues of the government accumulation equation are strictly positive. The equilibrium dynamics are:

$$db_t = [(\mathbf{r}(\Omega_t) - \phi_b)b_t - (r^* - \phi_b)b^*]dt \quad (82)$$

The first-order dynamics of this system around the steady-state are given by:

$$db_t = [\mathbf{r}(\Omega^*) - \phi_b + b^* \partial_b \mathbf{r}(\Omega^*)]dt \quad (83)$$

The last term is positive because of household behavior. The sum of the first two terms are positive under the condition $r^* > \phi_b$. This proves local uniqueness.

Global Uniqueness. We now show that explosive dynamics are incompatible with equilibrium. Appendix A.3 shows that a sufficient condition for explosive dynamics to be inconsistent with equilibrium is for real debt to grow at a rate greater than r_t . But this follows from Equation (82) and $\phi_b < 0$.

A.8 Unique Steady State with Real Rate Reaction Rule

Our argument for uniqueness proceeds in three steps. First, we show a real rate reaction rule can give rise to a unique steady-state. Second, we derive conditions for the steady-state to be saddle-path stable. This ensures local uniqueness. Finally, we consider whether explosive paths in debt can be ruled out globally. This ensures global uniqueness.

Steady-State Uniqueness. Suppose the government follows a fiscal rule of the form:

$$s_t = s^* + \phi_r(r_t - r^*) \quad (84)$$

where $s^* = r^*b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$db_t = [r_t b_t - s_t]dt \quad (85)$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{(b^* - \phi_r)r^*}{b - \phi_r} \quad (86)$$

Our goal is to obtain an upward sloping function for the null-cline that intersects the r -axis above $\mathbf{a}(r)$. This will ensure that it intersects the upwards sloping steady-state demand curve $\mathbf{a}(r)$ exactly once, as in Figure 6b. The slope of this function is

$$\frac{dr}{db} = -\frac{(b^* - \phi_r)r^*}{(b - \phi_r)^2} \quad (87)$$

which is strictly positive whenever $b^* > \phi_r$. We also want the null-cline to intersect the r -axis at a negative real interest rate that is greater than $\underline{r}$ (c.f. Figure 6b). This occurs if

$$\phi_r < \frac{s^*}{r^* - \mathbf{a}^{-1}(0)} \quad (88)$$

Local Uniqueness. We now examine conditions for this fiscal rule to give rise to local uniqueness. Under our maintained assumptions on the dynamical system that obtain (32), local uniqueness amounts to checking whether the eigenvalues of the government accumulation equation are strictly positive. The equilibrium dynamics are:

$$db_t = \mathbf{r}(\Omega_t)(b_t - \phi_r) - (r^* - \phi_r)b^*]dt \quad (89)$$

The first-order dynamics of this system around the steady-state are given by:

$$db_t = [\mathbf{r}(\Omega^*) + (b^* - \phi_r)\partial_b \mathbf{r}(\Omega^*)]dt \quad (90)$$

Note that at the top-right steady-state, we must have

$$\mathbf{r}'(\Omega^*) > -\frac{r^*}{b^*} \quad (91)$$

which ensures that a sufficient condition for the right-hand side of (90) to be positive is $\phi_r < 0$. Hence, $\phi_r < 0$ is a sufficient condition for local uniqueness.

Global Uniqueness. We now show that explosive dynamics are incompatible with equilibrium. Appendix A.3 shows that a sufficient condition for explosive dynamics to be inconsistent with equilibrium is for real debt to grow at a rate greater than r_t . But this follows from Equation (82) and $\phi_r < 0$.

A.9 Local Dynamics with Interest Payment Reaction Rule

Steady-State Invariance. Suppose the government follows a fiscal rule of the form:

$$s_t = s^* + \phi_s(r_t b_t - r^* b^*) \quad (92)$$

where $s^* = r^* b^*$ is consistent with any given point on the household demand curve, so that the tuple $(b^*, r^*) = (\mathbf{a}(r^*), r^*)$ with $r^* < 0$. The government accumulation equation is:

$$db_t = [r_t b_t - s_t] dt \quad (93)$$

The null-clines of the government accumulation equation are then defined by the following function:

$$r(b) = \frac{s^* - \phi_s r^* b^*}{b - \phi_s b^*} \quad (94)$$

$$= \frac{s^*}{b} \quad (95)$$

which shows that the steady-states are unchanged. Hence, there is no scope for this fiscal rule to eliminate steady-state multiplicity.

Local Dynamics. The dynamics of government debt are given by

$$db_t = (1 - \phi_s) (r(\Omega_t) b_t - s^*) dt \quad (96)$$

It follows that the stability properties of the two-steady states in the baseline case with $\phi_s = 0$ are reversed when $\phi_s > 1$.

A.10 Additional Details on the Wealth Distribution and MPCs

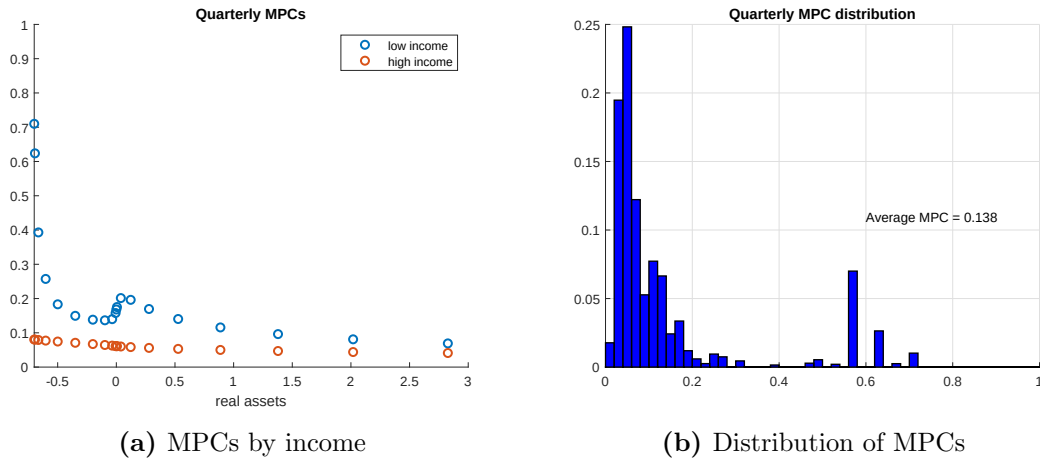


Figure 16: MPCs in the calibrated steady-state

This subsection provides some additional detail on the MPCs in the calibrated steady-state.

Figure 16a shows the dependence of marginal propensities to consume on real assets, disaggregated by the highest and lowest income draws. The plotted MPCs are the quarterly marginal propensities to consume from an unanticipated \$500 income gain.

MPCs are not monotonically decreasing in real assets because there is a borrowing wedge. Households with zero assets therefore have a high marginal propensity to consume because of the discontinuous cost of borrowing (Kaplan and Violante, 2014). Note that the MPCs of high income households lie uniformly below the MPCs of low income households.

Figure 16b plots the distribution of MPCs in the calibrated steady-state. A large number of households have an MPC of around 0.15 and hold zero assets. The average MPC in the economy is 0.14, which is in line with commonly estimated values for marginal propensities to consume (Jappelli and Pistaferri, 2010).

B Model Extensions

B.1 Household Problem with Diffusion Process

This section sets up an economy in which income follows a diffusion process. We derive as an auxiliary result that $r_t < \rho$ for all $t \geq 0$ in this economy.

Concretely, we assume that household income follows a diffusion process given by

$$dz_{jt} = \mu_z(z_{jt})dt + \sigma_z(z_{jt})dB_{jt} \quad (97)$$

where B_{jt} is adapted Brownian motion, independent across j , and $\mu_z(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma_z(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^+$ are twice-differentiable functions. We further assume that (97) admits a stationary distribution. The household problem now satisfies the following HJB equation:

$$\begin{aligned} \rho V_t(a, z) - \partial_t V_t(a, z) &= \max_c \frac{c^{1-\gamma}}{1-\gamma} + \partial_a V_t(a, z) [r_t a + z - \tau_t(z) - c] \\ &\quad + \mu_z \partial_z V_t(a, z) + \frac{1}{2} \sigma_z^2 \partial_{zz}^2 V_t(a, z), \end{aligned} \quad (98)$$

together with the boundary condition $\partial_a V_t(0, z) \geq (z - \tau_t(z))^{-\gamma}$. A solution to the HJB equation alongside (12) solves the household problem. The associated KFE equation is:

$$\partial_t g_t(a, z) = -\partial_a [g_t(a, z) \varsigma_t(a, z)] - \partial_z [\mu_z(z) g_t(a, z)] + \frac{1}{2} \partial_{zz}^2 [\sigma_z^2(z) g_t(a, z)] \quad (99)$$

Expected Consumption Dynamics. We now derive the expected consumption dynamics for unconstrained households. Following exactly the same steps outlined in Appendix A.1 for the case in which income follows a Poisson process, we can derive an Euler equation for unconstrained

households:

$$\begin{aligned}
(\rho - r_t)u'(c_t(a, z)) &= \mu_z(z)u''(c_t(a, z))\partial_z c_t(a, z) \\
&+ \frac{1}{2}\sigma_z^2(z) \left(u''(c_t(a, z))\partial_{zz}^2 c_t(a, z) + u'''(c_t(a, z))(\partial_z c_t(a, z))^2 \right) \\
&+ u''(c_t(a, z))[\partial_t c_t(a, z) + \varsigma_t(a, z)\partial_a c_t(a, z)]
\end{aligned} \tag{100}$$

We can also use Ito's lemma on $c_t(a_{jt}, z_{jt})$ to obtain

$$\begin{aligned}
dc_t(a_{jt}, z_{jt}) &= [\partial_t c_t(a_{jt}, z_{jt}) + \varsigma_t(a_{jt}, z_{jt})\partial_a c_t(a_{jt}, z_{jt})]dt \\
&+ [\mu_z(z_{jt})\partial_z c_t(a_{jt}, z_{jt}) + \frac{1}{2}\sigma_z^2(z_{jt})\partial_{zz}^2 c_t(a_{jt}, z_{jt})]dt + \sigma_z(z_{jt})\partial_z c_t(a_{jt}, z_{jt})dB_{jt}
\end{aligned} \tag{101}$$

Taking expectations of the above equation, combining it with (100), and imposing that u is isoelastic with curvature parameter γ yields the expected consumption dynamics for unconstrained households:

$$\frac{\mathbb{E}_t[dc_{jt}]}{c_{jt}dt} = \frac{1}{\gamma}(r_t - \rho) + \frac{\gamma + 1}{2}\sigma_z^2(z_{jt}) \left(\frac{\partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2 \tag{102}$$

Constrained households simply consume their income. Hence, their consumption dynamics are

$$dc_{jt} = [\mu_z(z_{jt})]dt + \sigma_z(z_{jt})dB_{jt} \tag{103}$$

The expected consumption dynamics of constrained households are therefore given by

$$\frac{\mathbb{E}_t[dc_{jt}]}{dt} = \mu_z(z_{jt}) \tag{104}$$

Derivation of Interest Rate Functional. Integrating over the consumption dynamics of unconstrained households and making use of the fact that

$$\int_j \frac{dc_{jt}}{dt} dj = 0$$

yields

$$\begin{aligned}
0 &= \int_{j:u} \frac{1}{\gamma}(r_t - \rho)c_{jt}dj + \int_{j:u} \frac{(\gamma + 1)}{2}c_t(a_{jt}, z_{jt}) \left(\frac{\sigma_z(z_{jt})\partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2 dj \\
&+ \int_{j:c} [\mu_z(z_{jt})c_t(a_{jt}, z_{jt})] dj
\end{aligned} \tag{105}$$

where we have used (102) and (104). Finally, imposing market clearing $\int_j c_{jt}dt = 1$ yields

$$r_t = \rho - \frac{\frac{\gamma(\gamma+1)}{2} \int_{j:u} c_t(a_{jt}, z_{jt}) \left(\frac{\sigma_z(z_{jt})\partial_z c_t(a_{jt}, z_{jt})}{c_t(a_{jt}, z_{jt})} \right)^2 dj + \gamma \int_{j:c} [c_t(a_{jt}, z_{jt})\mu_z(z_{jt})] dj}{1 - \int_{j:c} z_{jt}dj} \tag{106}$$

Note that this implies that $r_t < \rho$ for all $t \geq 0$ (not just in steady-state) if no households are constrained, or if $\int_{j:c} \mu_z(z_{jt}) dj > 0$, so that constrained households expect their income to increase, on average. We may also write the formula analogously as the one in the main text for the Poisson income process (26):

$$0 = \frac{\mathcal{C}_t^u}{\gamma}(r_t - \rho) + \mathcal{C}_t^u \tilde{\mathbb{E}}_t^u \left[\frac{\gamma + 1}{2} \sigma_z^2(z) \left(\frac{\partial_z c_t(a, z)}{c_t(a, z)} \right)^2 - \mu_z(z) \right] + \tilde{\mathbb{E}}_t [\mu_z(z)] \quad (107)$$

B.2 Model With Borrowing

In this section, we describe how the model is consistent with a non-zero lower bound on real household assets and costly borrowing. Households face a borrowing limit expressed in real terms:

$$\frac{A_{jt}}{P_t} \geq \tilde{a}_t \quad (108)$$

In order for the borrowing constraint to be consistent with balanced growth, we assume that $\tilde{a}_t$ grows at the rate of real output:

$$\tilde{a}_t = y_0 e^{g^t} \underline{a} \quad (109)$$

for some $\underline{a} < 0$. Note that this implies that

$$a_{jt} \geq \underline{a} \quad (110)$$

Furthermore, we assume that borrowing is costly. Households face a wedge $\vartheta \geq 0$ on the real interest rate when borrowing, so that the interest rate they pay on debt is $\vartheta + r_t$. We assume that the borrowing wedge creates deadweight loss in output that is equal to the wedge multiplied by the amount of assets borrowed. This implies that aggregate consumption is slightly less than output. The difference, however, is small and is around 0.3% of steady-state output in our main calibration. The associated boundary condition for the HJB equation (9) is:

$$\partial_a V_t(0, z) \geq (z - \tau_t(z) - (r_t + \vartheta)\underline{a})^{-\gamma} \quad (111)$$

The government accumulation equation continues to follow equation (18), with the understanding that $b_t \geq 0$, so that the government can lend, but not borrow.

B.3 Model With Long-Term Debt

We extend the model to long-term debt. We assume that the government now issues two securities: short-term debt B_t^s that pays a nominal rate i_t , and long-term debt B_t^l . We assume that long-term debt takes the form of depreciating consols that depreciate at a rate $\delta > 0$, and that yield a flow coupon payment of $\chi > 0$ as in [Cochrane \(2001\)](#). We let q_t denote the market value of this

long-term bond. The government's budget constraint can be written as:

$$dB_t^s + q_t dB_t^l = [iB_t^s + (\chi - \delta q_t)B_t^l - P_t s_t]dt \quad (112)$$

The intuition for this equation is as follows. The right-hand side is the government's nominal deficit that consists of the primary deficit $-P_t s_t$, interest payments on short-term debt iB_t^s , and coupon payments plus redemption of long-term debt $(\chi - \delta q_t)B_t^l$. Whenever the deficit is greater than zero, the government must issue additional debt. It can do so either by issuing additional short-term debt or by issuing additional long-term debt at the price of q_t .

Similarly, we may define the nominal short- and long-term debt holdings of household j at time t as A_{jt}^s and A_{jt}^l , respectively. The budget constraint of the household therefore becomes

$$dA_{jt}^s + q_t dA_{jt}^l = [i_t A_{jt}^s + (\chi - \delta q_t)A_{jt}^l + (z_{jt} - \tau_t(z_{jt}))P_t y_t - P_t \tilde{c}_{jt}]dt \quad (113)$$

We may define the market value of total government debt outstanding as:

$$B_t := B_t^s + q_t B_t^l \quad (114)$$

and the total value of household assets as:

$$A_{jt} := A_{jt}^s + q_t A_{jt}^l \quad (115)$$

We may also define de-trended real debt and assets as in the main text:

$$b_{jt} = \frac{B_t}{P_t} \quad \text{and} \quad a_{jt} = \frac{A_{jt}}{P_t} \quad (116)$$

The next proposition demonstrates that there is maturity structure irrelevance for government debt: b_t and a_{jt} are the only state variables in this economy.

Proposition 1. *The household budget constraint follows (6) and the real government budget constraint follows (18) for $t > 0$. Moreover, the price of long-term debt satisfies the following differential equation for $t > 0$:*

$$\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t \quad (117)$$

Proof. See Appendix E.4 □

This proof shows that an economy with long-term debt collapses into an economy with short-term debt in the absence of uncertainty. Equation (117) is an arbitrage relationship between short- and long-term debt. The left-hand side the sum of the capital gains due to changes in the price of long-term debt and the coupon payments. These must equal to the nominal interest rate. In equilibrium, households are indifferent between the two assets. Hence, long-term debt will only matter for inflation dynamics insofar there is an unanticipated change in nominal rates i_t . Equation

(117) is a forward-looking equation. Hence, in the absence of any nominal rate changes, real debt will always be equal to its steady-state value q^* , where:

$$q^* = \frac{\chi}{i^* + \delta} \quad (118)$$

where $i^* > 0$ is the constant value of the nominal rate.

B.4 Model With Foreign Demand for Debt

We assume that there exists a foreign sector that is populated by a representative household. The foreign representative household derives utility over real consumption streams and real debt holdings in terms of US goods.³⁸ Preferences over foreign consumption and bonds are given by

$$u(c_t, d_t) = \frac{c_t^{1-\gamma}}{1-\gamma} + \tilde{\zeta} \frac{d_t^{1-\theta}}{1-\theta}$$

with $\gamma \geq 0$ and $\theta \geq 0$. The parameter $\tilde{\zeta} > 0$ parameterizes the payoff derived from real bond holdings. Households' rate of time preference is $\tilde{\rho}$. We assume the foreign sector grows at the same rate g as the domestic economy, thereby allowing for the existence of a balanced growth path. The household's growth-adjusted discount rate is therefore $\rho := \tilde{\rho} - (1 - \gamma)g$. In addition, we define $r_t := i_t - \pi_t - g$.

The household's budget constraint in real and stationary terms is therefore

$$dd_t = [r_t d_t + y^f - c_t]dt \quad (119)$$

where $y^f > 0$ is the foreign household's endowment of the consumption good. Foreign real consumption and real debt holdings must satisfy the following Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\gamma} \left(r_t - \rho + \tilde{\zeta} \frac{d_t^{-\theta}}{c_t^{-\gamma}} \right)$$

In steady-state, the aggregate consumption of the foreign sector is y^f . Hence, for a given interest rate r^* , we must have

$$0 = r^* - \rho + \frac{\tilde{\zeta}}{y^f} d^{-\theta}$$

It follows that foreign sector demand for US government debt is given by

$$d(r^*) = \left(\frac{\rho - r^*}{\tilde{\zeta}} \right)^{-\frac{1}{\theta}}$$

³⁸Concretely, the foreign sector derives utility from nominal bonds in dollars divided by the US price level: $B_t/P_t^U S$. This is equivalent to holding real debt in terms of the foreign sector good ($P_t^F B_t/(P_t^F B_t)$) where P_t^F is the foreign price level. The implicit assumption here is that the final good is tradable, so that the exchange rate is constant.

where we let $\zeta := \tilde{\zeta}/y_f$. Note that the relationship between foreign debt holdings and real interest rates can be written as:

$$\log d = \zeta + \frac{1}{\theta} \log(r^* - \rho)$$

so a large θ means more inelastic demand.

Consider the limits of this function: as $r^* \rightarrow -\infty$, $d \rightarrow 0$ and as $r^* \rightarrow \rho$, $d \rightarrow \infty$. We want to argue that when the foreign demand is inelastic enough (θ large), we can get rid of the high inflation steady-state. From Figure 7, it is clear that what we need is the bond supply function $\mathbf{b}(r)$ to lie above the total bond demand function $\mathbf{a}(r) + \mathbf{d}(r)$ as $r \rightarrow -\infty$. This condition can be re-written as:

$$\lim_{r \rightarrow \infty} \frac{\mathbf{b}(r)}{\mathbf{a}(r) + \mathbf{d}(r)} < 1$$

Or, equivalently

$$\lim_{b \rightarrow 0} \frac{r^g(b)}{r^h(b) + r^f(b)} < 1$$

where $r^g(b)$, $r^h(b)$, and $r^f(b)$ denotes the inverse debt demand functions for the government, domestic households, and the foreign sector, respectively. Substituting for these functional forms yields

$$\lim_{b \rightarrow 0} \frac{\frac{s}{b}}{\mathbf{a}^{-1}(b) + (\rho - \zeta b^{-\theta})}$$

Appendix A.1 shows that there exists a finite real interest rate $\underline{r}$ such that $\mathbf{a}(\underline{r}) = 0$. We may apply L'Hopital's rule to obtain

$$\lim_{b \rightarrow 0} \frac{-s}{b^2(\kappa + \theta b^{-\theta-1})} < 1$$

where κ is a finite positive constant. This inequality is satisfied if and only if

$$-s < \theta b^{1-\theta}$$

Recall that $s < 0$ so the LHS is a positive finite number. As long as $\theta > 1$ the right-hand-side converges to infinity as $b \rightarrow 0$ which satisfies the inequality. Hence, the condition we need in order to obtain steady-state uniqueness is

$$\theta > 1$$

which implies that the foreign demand has to be inelastic enough.

B.5 Endogenous Output

In this subsection, we outline an economy in which labor is a variable input in production. Next, we discuss how endogenous output affects price level and inflation dynamics in response to unanticipated shocks.

B.5.1 Set-Up

Households. The set-up of the household problem closely follows that of the main text. However, we assume that households choose real consumption flows $\tilde{c}_{jt}$ and hours worked ℓ_{jt} to maximize

$$\mathbb{E}_0 \int e^{-\rho t} \left[\frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma} - \phi_t^{1-\gamma} \frac{\ell_{jt}^{1+\psi}}{1+\psi} \right] dt \quad (120)$$

where the expectation is taken with respect to households' efficiency units of labour z_{jt} . The exponent $\psi > 0$ is the inverse of the Frisch elasticity of labor supply. The term ϕ_t is a time-varying constant that augments the labor disutility in order to allow the economy to be consistent with balanced growth when $\gamma \neq 1$. Concretely, we assume that

$$\phi_t = \tilde{\phi} e^{gt} \quad (121)$$

where $\tilde{\phi} > 0$ and $g > 0$ is the growth rate of the economy. This formulation implies that a stationary equilibrium exists. Moreover, the distribution of hours across households is constant in the stationary equilibrium.³⁹ The households nominal budget constraint therefore satisfies

$$dA_{jt} = [i_t A_{jt} + (1 - \tau_{1t}) z_{jt} P_t w_t \ell_{jt} - P_t \tilde{c}_{jt} + P_t \tau_{0t}] dt \quad (122)$$

where w_t is the real wage rate for effective labor services at time t , τ_{0t} is a lump-sum payment and τ_{1t} is a constant proportional tax rate. We assume that τ_{0t} grows at a rate $g > 0$ in order to ensure that a stationary equilibrium exists:

$$\tau_{0t} = \tilde{\tau}_0 e^{gt} \quad (123)$$

Finally, the stochastic process for z_{jt} and the definition of de-trended real variables for the evolution of real debt are identical to those of the main text.

Firms. We assume that perfectly competitive firms hire labor to produce output y_t with the constant returns to scale (CRS) production function

$$y_t = \Theta_t L_t \quad (124)$$

³⁹We intentionally assume separability between hours and consumption in the instantaneous utility function so as to maximize comparability between the economy with endogenous output presented in this subsection and the endowment economy presented in the main text. In particular, the endowment economy can be closely approximated for large ψ and a given calibrated $\tilde{\phi}$. We note, however, that preferences by [King et al. \(1988\)](#) leave the key mechanisms unaffected.

where Θ_t is aggregate total factor productivity that grows at a rate $g > 0$ and L_t are total effective hours:

$$L_t := \int_j z_{jt} \ell_{jt} dj \quad (125)$$

CRS implies that the real wage rate w_t is equal to Θ_t for all $t \geq 0$.

Government. The dynamics for government debt are given by

$$dB_t = [i_t B_t - s_t P_t y_t] dt \quad (126)$$

where s_t is the ratio of primary surpluses to output and is determined by the $\tau_{0,t}$ and $\tau_{1,t}$ as

$$s_t = \frac{\tau_{0t}}{y_t} + \int_{j \in [0,1]} \tau_{1t} w_t z_{jt} \ell_{jt} dj \quad (127)$$

De-trended real government debt then follows

$$db_t = [r_t b_t - s_t] dt \quad (128)$$

We do not consider unanticipated changes in the nominal rate in this section. Consequently, we assume an interest rate peg $i_t = i^*$ without loss of generality in analyzing real dynamics.

Calibration. Our calibration sets $\psi = 2$, so that the intensive-margin Frisch elasticity of labor supply is equal to one-half, in line with the recommendation of [Chetty et al. \(2011\)](#). Moreover, we calibrate $\tilde{\phi}$ so as to set total hours worked equal to unity. Allowing labor to adjust on the intensive margin provides additional insurance to households. As such, the discount rate increases to 6.1% p.a. (relative to 2.8% p.a. from the calibration in the main text) in order to match a debt-to-annual GDP ratio of 1.10. The values for the remaining parameters remain unchanged from [Table 1](#).

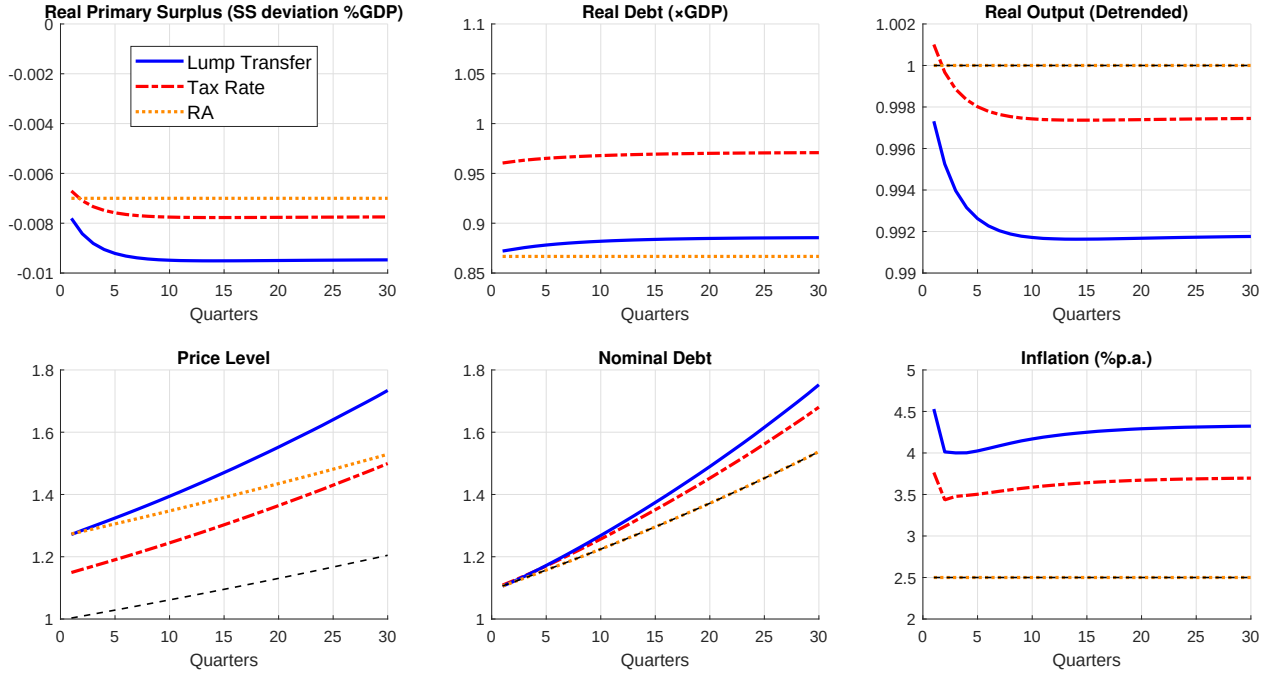
B.5.2 Quantitative Exercise

We consider the economy's response to an increase in deficits. First, we consider the economy's response to a permanent change in $\tilde{\tau}_{0t}$ from 0.333 to 0.340, keeping τ_1 fixed. Second, we consider a permanent change in τ_{1t} from 0.300 to 0.307, keeping τ_{0t} fixed. These changes amount to a change in deficits from 3.3% to 4% of GDP, if output was unchanged (in line with the analysis of [Section 5.5](#)).

An increase in deficits due to a tax cut results in a smaller jump in the initial price level, relative to the transfer expansion case. The main reason is that lower taxation increases the labor supply (whereas a transfer expansion lowers it). The corresponding rise in output raises tax revenues and attenuates the long-run increase in primary deficits relative to the transfer expansion.⁴⁰

⁴⁰The tax cut also increases precautionary motives by amplifying the volatility of post-tax earnings, in line with the reasoning of [Section 5.5](#). The real interest rate therefore decreases relatively less. Since the government now finances its debt at a higher cost, this a force that contributes to a larger initial jump in the price level. However, this mechanism is dominated by labor-supply channel.

Figure 17



Note: Impulse responses to a permanent expansion in primary deficits in the economy with endogenous output. The dotted orange line shows the effects of a permanent reduction in surpluses in the Representative Agent model due to a change in transfers. The solid blue line labelled “Lump Sum” illustrates the dynamics following an expansion of lump sum transfers. The dashed red line labelled “Tax Rate” illustrates the dynamics following a tax cut. In all experiments, deficits increase by 0.7% of pre-shock GDP.

In both economies, however, real output eventually *declines* relative to the representative agent benchmark. In order to understand this result, consider the tax cut experiment. There are two forces that contribute to an increase in labor supply. First, the tax cut directly raises the return to working, as explained above. Second, households in the new steady-state hold lower amounts of wealth, on average. This gives rise to positive wealth effects that also expands total hours worked. However, the new steady-state features a lower long-run real rate – a force only present in the heterogeneous agent economy. The reduction in the real rate increases consumption state-by-state due to the intertemporal savings motive, thereby reducing total hours worked. This last force is sufficiently strong that it counteracts the positive effect on output due to the lower tax rate and the change in the wealth distribution. Consequently, in the long-run output falls and deficits rise relative to the representative agent economy.

C Representative Agent Economy

This section outlines the representative agent economy, defines an equilibrium, and proves that there exists a unique real equilibrium when the government is running primary surpluses.

C.1 RA: Environment

Notation closely follows that of the main text. There exists a representative household that chooses real consumption flows $\tilde{c}_{jt}$ to maximize

$$\int e^{-\tilde{\rho}t} \frac{\tilde{c}_{jt}^{1-\gamma}}{1-\gamma} dt \quad (129)$$

Initial nominal assets A_0 are given. The household therefore faces a flow budget constraint

$$dA_t = [i_t A_t + (1 - \tau_t) P_t y_t - P_t \tilde{c}_t] dt \quad (130)$$

subject to the borrowing constraint $A_t \geq 0$, where τ_t is a path of taxes set by the government. We may express the budget constraint in real de-trended terms as

$$da_t = [r_t a_t + (1 - \tau_t) - c_t] dt \quad (131)$$

where the real rate is defined as:

$$r_t := i_t - \pi_t - g \quad (132)$$

Similarly, government debt evolves according to

$$db_t = [r_t b_t - \tau_t] dt \quad (133)$$

We also impose the commonly maintained assumption in the fiscal theory of the price level that the government can borrow, but not lend: $b_t \geq 0$.

Household Optimality. Given paths of real rates r_t and tax functions τ_t , the solution to the household problem must satisfy the following HJB equation:

$$\rho V_t(a) - \partial_t V_t(a) = \max_c \frac{c^{1-\gamma}}{1-\gamma} + \partial_a V_t(a) [r_t a + (1 - \tau_t) - c] \quad (134)$$

where we denote the growth-adjusted discount rate by $\rho = \tilde{\rho} - (1 - \gamma)g$. The borrowing constraint is again imposed via the boundary condition

$$(1 - \tau_t)^{-\gamma} = \partial_a V_t(0) \quad (135)$$

The first-order condition of this problem is:

$$c_t(a)^{-\gamma} = \partial_a V_t(a) \quad (136)$$

We also have the envelope condition:

$$\rho \partial_a V_t(a) - \partial_{ta}^2 V_t(a) = s_t(a) \partial_{aa}^2 V_t(a) + r_t \partial_a V_t(a) \quad (137)$$

where $s_t(a) := r_t a + (1 - \tau_t) - c$ denotes saving. Combining (136) and (137) yields:

$$\rho c^{-\gamma} + \gamma c_t(a)^{-\gamma-1} \partial_t c_t(a) = -\gamma s_t(a) c_t(a)^{-\gamma-1} \partial_a c_t(a) + r_t c^{-\gamma} \quad (138)$$

Dividing by $c^{-\gamma}$, and using the fact that:

$$\frac{dc_t(a_t)}{dt} = \partial_a c_t(a_t) s_t(a_t) + \partial_t c_t(a_t) \quad (139)$$

yields

$$\rho - r_t = -\gamma \frac{1}{c_t(a_t)} \frac{dc_t(a_t)}{dt} \quad (140)$$

which is the representative agent Euler equation in continuous time. Finally, the household's transversality condition is

$$\lim_{t \rightarrow \infty} e^{-\rho t} c^{-\gamma} a_t \leq 0 \quad (141)$$

Monetary Policy. We allow for arbitrary monetary policy rules i_t , but assume that they lead to well-defined paths for inflation given real rates r_t (see Section 2.3).

C.2 RA: Equilibrium Definition

We now define a real equilibrium for the representative agent economy.

Definition 2. *A real equilibrium is a collection of variables*

$$\{c_t, a_t, b_t, r_t\}_{t \geq 0} \quad (142)$$

such that:

1. For all $t > 0$, c_t satisfies the Euler equation (140) and transversality condition (141).
2. For all $t > 0$, a_t evolves according to the budget constraint (131).
3. For all $t > 0$, b_t evolves according to the government budget constraint (133).
4. For all $t \geq 0$, markets clear: $a_t = b_t$.

Note that by Walras' Law:

$$c_t = 1 \text{ for all } t \geq 0 \quad (143)$$

so that the goods market always clears.

C.3 RA: Uniqueness With Constant Surpluses

Next, we show that a real unique equilibrium exists whenever $\tau = \tau^* > 0$, so that the government is running constant surpluses.

First, note that the Euler equation (140) along with market clearing for output $c_t = 1$ implies

that $r_t = \rho$ for all $t \geq 0$. Integrating the government budget constraint forwards then yields (133):

$$b_0 = \lim_{T \rightarrow \infty} \left[\int_0^T e^{-\rho t} \tau^* dt + e^{-\rho T} b_T \right] \quad (144)$$

By transversality (141) and market clearing, the latter term must be non-positive. Moreover, it cannot be negative as this would violate the non-negativity constraint on household assets and/or the assumption that the government cannot be a lender. Hence, it must be zero. But this then implies that

$$b_0 = \lim_{T \rightarrow \infty} \left[\int_0^T e^{-\rho t} \tau^* dt \right] \quad (145)$$

$$b_0 = \frac{\tau^*}{\rho} \quad (146)$$

so b_0 is well-defined and strictly positive for any level of initial nominal assets B_0 . The dynamics for real debt $\{b_t\}_{t>0}$ are then pinned down by the government budget constraint (133). This proves the existence of a unique real equilibrium.

Given an initial level of nominal debt B_0 , uniqueness of the real equilibrium implies uniqueness of the initial price level P_0 . Subsequent inflation is uniquely pinned down by $r_t = \rho$ for all t , and a monetary policy rule which sets the path for the nominal rate i_t .

The Case of Deficits. The analysis above requires that the present discounted value in (145) be finite and positive. Hence, running persistent deficits cannot be an admissible equilibrium under the assumption that (i) households face borrowing constraints or (ii) that aggregate government debt must be non-negative.

D Representative Agent Economy with Bonds-In-Utility

This section formulates a representative agent economy augmented with bonds in the utility function (RA-BIU).

D.1 RA-BIU: Environment

Our notation follows closely that of the main text. Time is continuous and indexed by t . The economy is populated by a representative agent that derives utility from consumption streams c_t and real asset holdings a_t according to:

$$\int e^{-\rho t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + \zeta \ln(a_t + \underline{a}) \right) dt \quad (147)$$

where $\rho > 0$ denotes the household's discount rate, and $\zeta, \underline{a}$ are positive constants. Our assumption that real assets enter utility in a logarithmic fashion is inessential to the main results. However, logarithmic utility will allow us to characterize the steady-states of the economy in closed-form.

The households initial nominal assets $A_0 = P_0 a_0$ are given. For $t > 0$ the household faces the flow budget constraint

$$dA_t = [i_t A + (1 - \tau_t) P_t y_t - P_t c_t] dt \quad (148)$$

where i_t denotes the nominal rate on assets, τ_t is the tax rate on income, and y_t is real aggregate output at time t . To simplify the exposition, in this section only, we assume zero growth. Positive growth is straightforward to incorporate by letting ζ grow over time at the appropriate rate, as we have explained in the model with flexible labor supply of Section B.5.

At all points in time in which the price level is differentiable, the household budget constraint can be written in real terms as follows:

$$da_t = [r_t a_t + (1 - \tau_t) y_t - c_t] dt \quad (149)$$

where $r_t = i_t - \pi_t$ denotes the real interest rate on bonds. The government budget constraint can similarly be written in real terms as:

$$db_t = [r_t b_t - \tau_t y_t] dt \quad (150)$$

We also impose the commonly maintained assumption in the fiscal theory of the price level that the government can borrow, but not lend: $b_t \geq 0$.⁴¹

Household Optimality. The representative household takes the future sequence of real rates r_t and output y_t as given, and chooses consumption and real asset holdings optimally subject to its budget constraint (149). This implies the following Euler Equation

$$\frac{1}{c_t} \frac{dc_t}{dt} = \frac{1}{\gamma} \left(r_t - \rho + \frac{\zeta c_t^\gamma}{a + \underline{a}} \right) \quad (151)$$

Moreover, the household must satisfy the following transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\rho t} c_t^{-\gamma} a_t \leq 0 \quad (152)$$

D.2 RA-BIU: Equilibrium Definition

We now define an equilibrium in real terms, under the assumption that the price level is differentiable for all $t > 0$.

Definition 3. *A real equilibrium is a collection of variables*

$$\{c_t, a_t, b_t, r_t\}_{t \geq 0} \quad (153)$$

such that:

⁴¹As explained in the context of the RA model of Section C, this can also be rationalized through a borrowing constraint on the household side.

1. For all $t > 0$, c_t satisfies the Euler equation (151) and transversality condition (152).
2. For all $t > 0$, a_t evolves according to the budget constraint (149).
3. For all $t > 0$, b_t evolves according to the government budget constraint (150).
4. For all $t \geq 0$, markets clear: $a_t = b_t$.

Price Level Determination. As in the main text, each real equilibrium defines a unique price level determined by:

$$P_0 = \frac{B_0}{b_0} \quad (154)$$

The path of inflation is then determined residually through the Fisher identity

$$\pi_t = i_t - r_t \quad (155)$$

We assume for simplicity a monetary policy peg, $i_t = i^*$, but note that all our results on equilibrium uniqueness extend to the more general monetary rules outlined in Section 2.3.

D.3 RA-BIU: Uniqueness with Constant Surpluses

We now show that a unique real equilibrium exists under a constant, strictly positive surplus rule $\tau_t = \tau^*$, where $\tau^* > 0$.

Proposition 2. *A unique real equilibrium exists. Moreover, $r_t = r^*$ and $b_t = b^*$ for all $t \geq 0$, where r^* and b^* are strictly positive constants that are given by:*

$$r^* = \begin{cases} \frac{-(\tau^* + \zeta - \rho \underline{a}) + \sqrt{(\tau^* + \zeta - \rho \underline{a})^2 + 4\rho \underline{a} \tau^*}}{2\underline{a}} & \text{if } \underline{a} > 0 \\ \frac{\rho \tau^*}{\tau^* + \zeta} & \text{if } \underline{a} = 0 \end{cases} \quad (156)$$

and

$$b^* = \frac{\tau^*}{r^*} \quad (157)$$

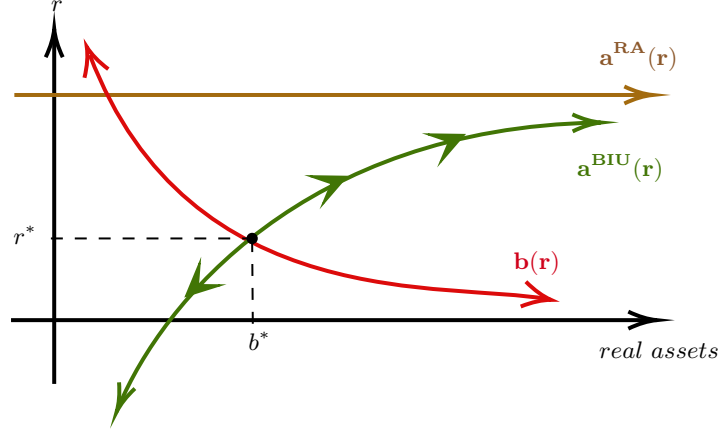
Proof. See Appendix E.1 □

The intuition for this result closely mirrors that of the representative agent economy. The system of equations (150) and (151) are globally unstable. Paths in which $b_0 < b^*$ therefore lead to downward explosions, which violate the non-negativity condition on debt. Paths in which $b_0 > b^*$ lead to an excessive accumulation of assets, thereby violating household optimality. These dynamics are graphically depicted in Figure 18. Note that r^* is strictly increasing in τ^* , with $r^* \rightarrow \rho$ and $a^* \rightarrow \infty$ as $\tau^* \rightarrow \infty$. In this sense, the steady-state asset demand in the RA-BIU economy (169) exhibits many similar features to the heterogeneous agent economy considered in the main text.

D.4 RA-BIU: Dynamics with Constant Deficits

Next, we consider dynamics under constant deficits $\tau^* < 0$. We show that the price level is generally no longer determinate for a given value of initial nominal debt. Intuitively, the steady-states of the

Figure 18



Note: This figure plots the dynamics for real assets in the RA-BIU economy when $\tau^* > 0$, as given by (150) and (151)

government accumulation equation (150) form an *upward* sloping locus in $r - b$ space, as depicted graphically in Figure 19. This can give rise to steady-state multiplicity, eliminating the explosive dynamics that are required in order to obtain uniqueness. The following proposition formally characterizes the nature of this steady-state multiplicity.

Proposition 3. *Suppose $\rho \underline{a} < \zeta$. Then:*

1. *If $\underline{a} = 0$, a unique steady-state exists if $\tau^* \in (-\zeta, 0)$, and no steady-state exists if $\tau^* \in (-\infty, -\zeta]$.*
2. *If $\underline{a} > 0$, there exists a $\underline{\tau} \in (\rho \underline{a} - \zeta, 0)$ such that two distinct steady-states exist if $\tau^* \in (\underline{\tau}, 0)$, no steady-state exists if $\tau^* \in (-\infty, \underline{\tau})$, and a unique steady-state exists if $\tau^* = \underline{\tau}$*

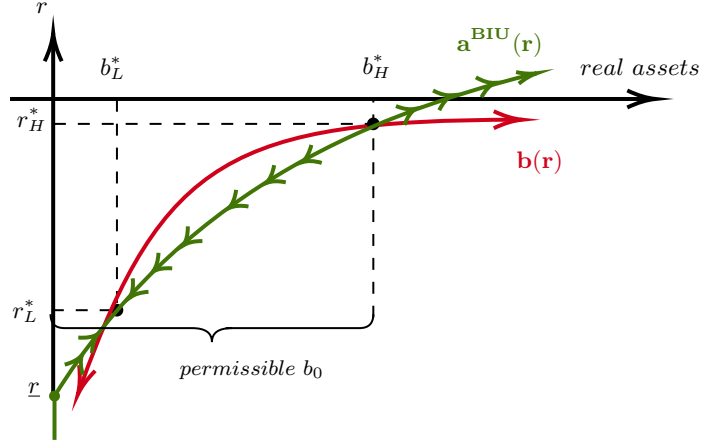
Proof. See Appendix E.2. □

The condition $\rho \underline{a} < \zeta$ ensures that there exists a negative interest rate such that households are willing to hold strictly positive amounts of real assets (no steady-state with deficits exists trivially if this condition is not satisfied). Note also that, depending on the value of $\underline{a}$, zero, one, or two equilibria can exist. Further, at least one equilibrium exists as long as the level of deficits is not too large. We next show how steady-state multiplicity is tied to price level determinacy. In particular, a unique equilibrium exists if and only if a unique steady-state exists.

Proposition 4. *The following statements are true.*

1. *If no steady-state exists, then no real equilibria exist.*
2. *If a unique steady-state exists, then there exists a unique real equilibrium with constant real rates $r_t = r_H^* = r_L^*$ and real assets $b_t = b_H^* = b_L^*$.*
3. *If two distinct steady-states exist, then there exists a continuum of real equilibria indexed by $b_0 \in (0, b_H^*]$.*

Figure 19



Note: This figure plots the dynamics for real assets in the RA-BIU economy for $\tau^* < 0$, as given by (150) and (151).

Proof. See Appendix E.3. □

One can show that the presence of two steady-states imply a non-singular basin of attraction for the economy. Hence, a continuum of real equilibria, indexed by their initial condition b_0 , are possible. Note that the final condition places a lower bound on the price level for any given level of initial nominal assets, given by $P_0 = \frac{B_0}{b_H^*}$.

E Omitted Derivations and Proofs

E.1 Proof of Proposition 2

Proof. Our proof proceeds in several steps.

Step 1: Monotonicity of real assets. We first show that a_t is increasing if $a_t > a^*$ and decreasing if $a_t < a^*$, where $a^* > 0$ is the unique steady-state value of real debt. We then show that this implies that $a_t < a^*$ at any t violates the non-negativity constraint on debt. Finally, we show that $a_t > a^*$ at any t is inconsistent with household optimality.

First, note that equation (151) together with $c_t = 1$ at all t implies that the real rate is given by the following equation for all t

$$r_t = \rho - \zeta \frac{1}{a_t + \underline{a}} \quad (158)$$

Imposing market clearing and using the government budget constraint (150), we can derive an expression for the dynamics of real debt

$$\dot{a}_t = \left(\rho - \zeta \frac{1}{a_t + \underline{a}} \right) a_t - \tau^* \quad (159)$$

where $\dot{a}_t \equiv \frac{da_t}{dt}$. The steady-states of this differential equation are given by

$$\frac{\tau^*}{a^*} = \rho - \zeta \frac{1}{a^* + \underline{a}} \quad (160)$$

Note that the left-hand side of the above equation is decreasing in a^* whenever $\tau^* > 0$ (and asymptotes to zero as $a^* \rightarrow \infty$ and infinity as $a^* \rightarrow 0$), while the right-hand side is increasing in a^* (and asymptotes to $\rho > 0$ as $a^* \rightarrow \infty$). Moreover, both terms are continuous for $a^* > 0$. Hence, a unique steady-state with a strictly positive real rate exists. Denote this real rate by $r^* > 0$.

Further, $\dot{a}_t$ is strictly positive whenever $a_t \in (a^*, \infty)$ and strictly negative whenever $a_t \in (0, a^*)$. Suppose otherwise. We have that:

$$\frac{d\dot{a}_t}{da} \Big|_{a_t=a^*} = r^* + \zeta a^* (a^* + \underline{a})^{-2} > 0 \quad (161)$$

Moreover, $\dot{a}_t$ is continuously differentiable on $a_t > 0$. Hence, $\dot{a}_t(a_t) < 0$ for some $a_t \in (a^*, \infty)$ would imply that there exists an $a^{**} \in (a^*, \infty)$ such that $\dot{a}_t(a^{**}) = 0$, thereby violating steady-state uniqueness.

Step 2: Ruling Out Downwards Explosions. Next, we rule out all equilibria in which $a_{t'} < a^*$ for some $t' \geq 0$. By Step 1, $a_{t'} < a^*$ implies that $a_{t'} < a_t$ for all $t' \geq t$. Moreover, there are no limit points such that $\lim_{t \rightarrow \infty} a_t = a^{**}$ for any $a^{**} > 0$. Hence, any path in which $a_{t'} < a^*$ implies that the constraint $a_t > 0$ must be violated in finite time.

Step 3: Ruling Out Upwards Explosions. We now rule out equilibria in which $a_{t'} > a^*$ for

some $t' \geq 0$. We may integrate the government budget constraint (150) forwards to obtain

$$a_0 = \lim_{T \rightarrow \infty} \left[\int_0^T \exp \left(\int_0^s -r_u du \right) \tau^* ds + \exp \left(\int_0^T -r_u du \right) a_T \right] \quad (162)$$

Note that $r_t > 0$ whenever $a_t > a^*$, so the first term in the limit is well-defined. Moreover, (150) implies that assets will be growing at rate r_t whenever $a_t \geq a^*$. Hence, the second-term is non-zero if and only if $a_{t'} \geq a^*$ for some $t' \geq 0$.

We now show that household optimality implies that this second term must necessarily be finite. Substituting for the real rate, we obtain:

$$\lim_{T \rightarrow \infty} \left[\exp(-\rho T) a_T \times \exp \left(\int_0^T \zeta \frac{1}{a_u + \underline{a}} du \right) \right] \quad (163)$$

The first-term in this expression is zero by the transversality condition (152). The second term is bounded as assets are growing at an exponential rate. Hence, we must have $a_t = a^*$ and $r_t = r^*$ for all $t \geq 0$. Equation (162) then implies that the second term is zero and thus a_0 must be given by

$$a_0 = \int_0^\infty \exp \left(\int_0^s -r^* du \right) \tau^* ds \quad (164)$$

$$= \frac{\tau^*}{r^*} \quad (165)$$

$$= a^* \quad (166)$$

Substituting for a^* in (158) yields a quadratic equation with a unique, strictly positive root given by (156). This completes the proof. $\square$

E.2 Proof of Proposition 3

Proof. We may substitute for steady-state assets in (158) to obtain

$$r^* = \rho - \frac{\zeta}{\frac{\tau^*}{r^*} + \underline{a}} \quad (167)$$

We may solve the above equation to express the steady-states of the system as:

$$r^* = \begin{cases} \frac{-(\tau^* + \zeta - \rho \underline{a}) \pm \sqrt{(\tau^* + \zeta - \rho \underline{a})^2 + 4\rho \underline{a} \tau^*}}{2\underline{a}} & \text{if } \underline{a} > 0 \\ \frac{\rho \tau^*}{\tau^* + \zeta} & \text{if } \underline{a} = 0 \end{cases} \quad (168)$$

and

$$a^* = \frac{\tau^*}{r^*} \quad (169)$$

where we additionally require $r^* < 0$ so that the non-negativity constraint on assets is not violated. It is straightforward to see that this condition is satisfied if and only if $\tau^* > -\zeta$ when $\underline{a} = 0$. This

proves the first part of the proposition.

To prove the second part of the proposition, note a necessary and sufficient condition for $r^* < 0$ in the constant deficit economy is $\tau^* \in (\rho\underline{a} - \zeta, 0)$ and that $(\tau^* + \zeta - \rho\underline{a})^2 + 4\rho\underline{a}\tau^* > 0$. This is negative at $\tau^* = \rho\underline{a} - \zeta$, positive at $\tau^* = 0$, and strictly increasing on $(\rho\underline{a} - \zeta, 0)$. Hence, there exists a unique root of this expression within this interval given by $\underline{\tau} \in (\rho\underline{a} - \zeta, 0)$. It follows that there are two distinct steady-states whenever $\underline{\tau} < \tau^* < 0$, no steady-states whenever $\tau^* < \underline{\tau}$ and a unique steady-state whenever $\tau^* = \underline{\tau}$. $\square$

E.3 Proof of Proposition 4

Proof. Suppose no steady-states exist. Equation (150) then implies that real assets will tend to infinity or minus infinity for any given b_0 . The former case is ruled out, as it violates the transversality condition by Proposition 2. The latter case is ruled out as it implies that assets will violate their non-negativity constraint in finite time. Hence, no equilibria exist.

Next, suppose that a unique steady-state exist. Define the function

$$r(a_t) = \rho - \frac{\zeta}{a_t + \underline{a}} \quad (170)$$

From (150), steady-states are given by the roots to

$$g(a) = r(a) - \frac{\tau^*}{a} \quad (171)$$

There exists a unique a^* such that $g(a^*) = 0$ by assumption. Moreover, $g(a) \rightarrow \rho > 0$ as $a \rightarrow \infty$, so we must have $g'(a^*) > 0$ by the intermediate value theorem. Using the government accumulation equation, the dynamics of real debt around a^* are given by

$$\begin{aligned} d\frac{\tilde{a}_t}{dt} &= [r'(a^*)a^* + r(a^*)]\tilde{a}_t \\ &= [r'(a^*) + \frac{\tau^*}{(a^*)^2}]\frac{\tilde{a}_t}{a^*} \\ &= g'(a^*)\frac{\tilde{a}_t}{a^*} \\ &> 0 \end{aligned}$$

where $\tilde{a}_t = a_t - a^*$, to first-order. Because a^* is unique by assumption, real assets explode upwards exponentially at a rate r_t when $a_0 > a^*$ (violating (152)) and downwards when $a_0 < a^*$ (violating the non-negativity of assets in finite time). Hence, a unique equilibrium exists.

Suppose now that two equilibria exist $a_H^* > a_L^*$. The top equilibrium is locally unstable by the argument presented above. The bottom equilibrium is locally stable, since $g(a) \rightarrow \infty$ as $a \rightarrow \infty$. Hence, $g'(a_L^*) < 0$. This implies that all equilibria with $b_0 \in (0, b_H^*)$ converge to b_L^* , while all equilibria with $b_0 > b_H^*$ feature explosive dynamics that violate (152). Thus, there exist a continuum of equilibria indexed by $b_0 \in (0, b_H^*]$. $\square$

E.4 Proof of Proposition 1

Proof. We define the auxiliary variable $u = A_{jt}^l$. Note that this implies $dA_{jt}^l = u$. Hence, the households HJB equation is given by:

$$\begin{aligned} & \tilde{\rho}V_t(A^l, A^s, z) - \partial_t V(A^l, A^s, z) = \\ & \max_{c,u} \frac{c^{1-\gamma}}{1-\gamma} + \tilde{s}_t \partial_{A^s} V_t(A^l, A^s, z) + \partial_{A^l} V_t(A^l, A^s, z)u + \sum_{z' \neq z} \lambda_{zz'} [V_t(A^l, A^s, z') - V_t(A^l, A^s, z)] \end{aligned}$$

where

$$\tilde{s}_t := i_t A^s + (\chi - \delta q_t) A^l + (z - \tau(z)) P_t y_t - P_t \tilde{c}_t - q_t u$$

The first-order condition with respect to u is given by:

$$q_t \partial_{A^s} V(A^l, A^s, z) = \partial_{A^l} V_t(A^l, A^s, z) \quad (172)$$

We may differentiate with respect to time to obtain:

$$q_t \partial_{A^s, t}^2 V_t(A^l, A^s, z) + \partial_t q_t \partial_{A^s} V(A^l, A^s, z) = \partial_{A^l, t} V_t(A^l, A^s, z) \quad (173)$$

The envelope condition for the HJB with respect to A^l is:

$$\tilde{\rho} \partial_{A^l} V_t - \partial_{t, A^l}^2 V_t = \tilde{s}_t \partial_{A^s, A^l}^2 V_t + (\chi - \delta q_t) \partial_{A^l} V_t + u \partial_{A^l}^2 V_t + \sum_{z' \neq z} \lambda_{zz'} [\partial_{A^l} V_t - \partial_{A^l} V_t] \quad (174)$$

□

Similarly, the envelope condition for the HJB with respect to A^s is:

$$\tilde{\rho} \partial_{A^s} V_t - \partial_{t, A^s}^2 V_t = \tilde{s}_t \partial_{A^s}^2 V_t + i_t \partial_{A^s} V_t + u \partial_{A^l, A^s}^2 V_t + \sum_{z' \neq z} \lambda_{zz'} [\partial_{A^s} V_t - \partial_{A^s} V_t] \quad (175)$$

Multiplying (175) by q_t , subtracting Equation (174) from (175) and using (172) and (173) yields:

$$(q_t i_t - (\chi - \delta q_t) - \partial_t q_t) \partial_{A^l} V_t = 0 \quad (176)$$

By market clearing, we must have $\partial_{A^l} V_t > 0$ (otherwise no long-term debt would be purchased in equilibrium). Hence, we have the arbitrage relationship:

$$\frac{\dot{q}_t}{q_t} + \frac{\chi - \delta q_t}{q_t} = i_t \quad (177)$$

Differentiating $B_t = q_t B_t^l + i_t B_t^s$ and using the (112) yields (15), which can be written in real terms. This completes the proof.