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Internal Adjustment Costs of Firm-Specific Factors and the Neoclassical Theory of the Firm

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Abstract

This paper considers the consequences of a two-sector vertically-integrated model of firms producing output using firm-specific capital with a second sector producing firm-specific capital by adapting raw capital purchased in the market. Analysts rarely observe each sector separately. Aggregating over both sectors produces short-run and long-run factor demand functions that appear to be perverse, but when disaggregated obey standard neoclassical properties. Adjustment costs create the appearance of static inefficiency in the presence of dynamic efficiency.

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Peter Schmidt has made important contributions to production theory in his work on estimating efficiency frontiers (Schmidt, 1976, 1978; Schmidt and Lovell, 1979, 1980). He surveys this work in his illuminating survey with Førsund et al. (1980). Central to this work is the assumption of efficient production characterized by static neoclassical factor demand functions with standard properties.

The standard model ignores adjustment costs and the presence of firm-specific factors of production. This paper considers the case where firms can buy raw capital goods or raw labor in the market, but then tailor these inputs to the particular production processes of the firm. We thus account for firm-specific human capital as discussed in Becker (1964). Firms can also buy machines in the capital goods market and then mold them to its specific production processes.

This paper discusses how these considerations alter the theory of the firm, including the standard properties of derived factor demand functions. In two influential papers, Treadway (1970, 1971) develops the theory of the firm when there are internal adjustment costs so that installation of new capital initially reduces the output of the firm so that the marginal cost of a unit of new capital is the market price of raw goods plus marginal installation costs. Treadway ignores the specificity of the installed capital, as does a large ensuing literature (e.g., McLaren and Cooper, 1980).

Treadway develops the theory of demand for inputs in the presence of adjustment costs and presents results that appear, at first sight, to contradict neoclassical theory. We investigate his model and show that any apparent contradictions are the result of an aggregation problem. The firm has two sectors: (a) a final output producing sector, and (b) a capital finishing sector that transforms raw inputs purchased in the market and tailors them to the requirements of the output sector. Aggregating these sectors into a single synthetic firm is the source of apparent contradictions with standard neoclassical theory. Thus, in the short-run, an increase in the price of labor can increase the demand for labor to purchase labor for use in the capital input building sector to facilitate substitution for labor in the final goods sector. In the long run, labor demand can fall even if capital and labor are complements in each sector. Short-run demand can also be more elastic than long-run demand, contrary to intuitions based on the Le Chatelier principle. The presence of adjustment costs gives rise to the appearance of static inefficiency, when in fact, the firm is operating efficiently.[1]

Enroute to establishing these results, we also investigate the economic foundations of the Koyck distributed lag model to which Schmidt has made notable contributions (Schmidt, 1974, 1975; Schmidt and Waud, 1973). Baltagi (2008) reviews this literature.

1 Introduction

Adjustment costs play important roles in empirical economics as explanations of delays in responses to economic stimuli. Koyck (1954) formalized the adjustment cost model for investment and his model is widely used, although not always cited. A recent example is Chetty and Szeidl (2016). Eisner and Strotz (1963) derived an economic model justifying it. Economists have used these models to analyze the dynamic behavior of firms (Lucas, 1967; Nerlove, 1972; Treadway, 1970). They suggest extensions of the simple Eisner and Strotz model to account for "internal" or "nonseparable" adjustment costs. In such mod-

[1]This point is extensively developed in Silva et al. (2020), who base their analysis on the Treadway (1970) model. See also Tsionas et al. (2020).

els the cost of adjustment of one quasi-fixed factor is influenced by freely variable factors of production. The notion of nonseparable costs is sometimes presented as a generalization of the neoclassical production function. The literature (especially Treadway, 1970) demonstrates that many of the familiar theorems of comparative statics of factor defined are not valid.[2] The thrust of this literature is that accounting for dynamic adjustment costs alters the conventional theory of the firm. For example, it is not possible to demonstrate (a) the non positivity of own-price effects on factor demands, (b) the symmetry of cross-price effects for factor demands, or (c) that short-run effects are less elastic than long-run effects.

This paper investigates the aggregation problem that leads to such results and apparently contradicts neoclassical production theory. We clarify these claims and draw out the implications of dynamics for the neoclassical theory of production and producer demand. The literature on adjustment costs suggests that the firm can be viewed as producing two outputs—final product and installed inputs, capital or human capital, or both.[3] However, the discussion in the literature is reduced form in nature and the two production processes are not distinguished.

Any novelty in the adjustment cost literature in regard to factor demand does not arise from the dynamics but in aggregating two distinct sectors. Disaggregating them eliminates any conflict with standard neoclassical theory.

[2]Hicks (1932) is one statement of the standard theories.

[3]See, e.g., Lucas (1967); McLaren and Cooper (1980); Nerlove (1972); Treadway (1970).

2 The Model

Following Treadway (1970) and the subsequent literature, we assume that the firm produces a single product, using two factors of production, capital, K, and labor, L. This simple model is easily generalized to the vector case, but for the sake of simplicity, we do not do so. The firm is assumed to maximize its present value:

$$V = \int_0^\infty [PQ - wL - gI]e^{-rt}dt, \tag{1}$$

where Q is output, p is its price, w is the price of labor and g is the price of investment goods. This objective function generalizes the profit maximization hypothesis used in static models. Capital is accumulated via:

$$\dot{K} = I - \mu K \tag{2}$$

$$K(0) = K_0. \tag{3}$$

μ is the rate of depreciation. Treadway's generalized (dynamic) final output production function is

$$Q = F(K, L, \dot{K}). \tag{4}$$

F is assumed to be concave and twice continuously differentiable. Equations 1–4 complete the specification of the model. Lucas (1967) and Nerlove (1972) use versions of this model as well as do later papers. We dig more deeply into the interpretation of Equation 4.

In the theory of capital and growth, the rate of investment is sometimes included in the production function to reflect the fact that technological progress is embodied in the investment in capital goods. In contrast, the adjustment models considered here assume that net investment decreases current output. This is due to planning costs, break-in costs, and the like. While these costs undoubtedly occur, there is no reason why they should be amalgamated into the production function, which reflects a purely technical relationship, instead of treating them as additional costs as do Eisner and Strotz.

Nerlove and Lucas argue that the firm could be viewed as producing its "capital in place" in addition to the final product. In this case, equation (4) has an economic interpretation as the transformation curve or the production function for two outputs using two inputs.

However, a difficulty arises in the interpretation of differential equation (2) for the capital stock. After generalizing the production function, this literature uses the same differential equation for capital stock used in dynamic models with less general production functions. If the firm is producing its "capital stock in place," the change in capital stock in place cannot be simply equal to the quantity of capital bought in the market place.

Production of capital stock in place leads to some loss in current output. Since labor is assumed to be variable in standard neoclassical models, this must imply that capital goods are used in their own production. Thus, the production function for new capital must involve some existing capital as an argument in addition to the purchased inputs and labor. In other words, the relations (2) and (4) are mathematically inconsistent. Hence, the equation for $\dot{K}$ should be modified to proceed with the analysis.

We modify (2) and (4). We assume total capital K can be allocated to either sector $(K_1 + K_2 = K)$ and labor may be employed in building finished capital (L_1) or in producing finished output. Thus,

$$\dot{K} = f(K - K_1, L_2, I) - \mu K, \qquad K_0 \leq K_1 \leq K \tag{2'}$$

$$Q = F(K_1, L_1) \tag{4'}$$

Both f and F are assumed to be concave and twice continuously differentiable and satisfy Inada conditions.

In this model, the firm's capital stock is like an "ectoplasm" (Robinson, 1964), which can be used as an input in the product sector or in the machinery sector. It is not clear why some of the machines that are already in place should be dismantled to put the new ones in place. However, we retain this assumption to make the model similar to the cited literature. Hence, aggregate L is defined as $L_1 + L_2$. We also assume that there is no secondhand market for installed capital goods.[4] These assumptions are captured by equation (2′). We make these assumptions not because they are realistic but simply to be consistent with the hypothesis that the firm foregoes some current output to provide new capital.

It should now be apparent that the model we have proposed is more like a two-sector optimal growth model as analyzed by Uzawa (1964) and Srinivasan (1964), than the standard neoclassical model of a firm in a competitive industry. Unlike Treadway (1970) and the subsequent literature we explicitly model the source of adjustment costs. We next

[4] It will be messy to allow for the possibility of selling the finished capital or buying it on the market. This assumption is, however, invoked in later work by McLaren and Cooper (1980) who consider duality in a model closely following Treadway (1970) and using his specification without an explicit treatment of adjustment costs. A more general approach would introduce another production function for dismantling the capital in place. In our model, if the firm has larger "ectoplasm" in the product sector than the desired stock, it has to close down its "ectoplasm" producing branch for a while and reach a steady state. Such models are perhaps more appropriate to analyze the behavior of the firm when certain factors like human capital are internally produced.

demonstrate how the tools of comparative statics and the optimal control theory can be used to derive some implications for the long and the short run consistent with neoclassical theory, contrary to claims in the cited literature.

3 Comparative Statics

In this section, we analyze the changes in the long-run equilibrium values of capital and labor due to changes in the exogenous variables like the wage rate. We assume that the firm is in its long-run equilibrium with no incentive to change its capital stock, i.e., when $\dot{K} = 0$.

Let the price of the product be P. There is no market price for finished capital goods, since they are internally produced, but there is a price for the raw input I. Let λ be the long-run shadow price of capital, which the firm values.[5] The flow price capital is $(\rho + \mu)\lambda$ and is not in general g, the price of investment goods. The following conditions are necessary for maximizing profit.[6] In the production sector,

$$PF_{K_1} = (\rho + \mu)\lambda \tag{5}$$

$$PF_{L_1} = w \tag{6}$$

[5] See, e.g., Arrow and Kurz (1970).

[6] These conditions are obtained by maximizing (1) subject to (2′), (3), and (4′).

In the machinery producing sector, the following optimality conditions hold:

$$\lambda f_K = (\rho + \mu)\lambda \tag{7}$$

$$\lambda f_{L_2} = w \tag{8}$$

$$\lambda f_I = (\rho + \mu)g. \tag{9}$$

Since net investment is zero in the long run, from (2′), we obtain:

$$f_{K-K_1,L_2,I} = \mu k. \tag{10}$$

These six equations in six unknowns can be solved to obtain the long-run demand for capital and labor as a function of the product and factor prices. Second order conditions for optimality follow from standard concavity and Inada conditions. To determine the changes in the long-run demand for factors due to changes in exogenous variables, we differentiate the equations (5)–(10) totally and, observing that $dK = dK_1 + dK_2$, to obtain,

$$\begin{bmatrix} PF_{K_1K_1} & PF_{K_1}L_1 & 0 & 0 & 0 & \rho+\mu \\ PF_{L_1K_1} & PF_{L_1L_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda f_{KK} & \lambda f_{KL_2} & \lambda f_{KI} & 0 \\ 0 & 0 & \lambda f_{L_2K} & \lambda f_{L_2L_2} & \lambda f_{L_2I} & f_{L_2} \\ 0 & 0 & \lambda f_{IK} & \lambda f_{IL_2} & \lambda f_{II} & f_I \\ -f_K & 0 & f_K - \mu & f_{L_2} & f_I & 0 \end{bmatrix} \begin{bmatrix} dK_1 \\ dL_1 \\ dK_2 \\ dL_2 \\ dI \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda d(\rho+\mu) \\ dw \\ \lambda d(\rho+\mu) \\ dw \\ gd(\rho+\mu) \\ 0 \end{bmatrix} \tag{11}$$

where the subscripts show the arguments with respect to which the function is differentiated. Denote the matrix on the left-hand side of (11) by H. Assuming H is invertible,

$$\begin{bmatrix} dK_1 \\ dL_1 \\ dK_2 \\ dL_2 \\ dI \\ d\lambda \end{bmatrix} = H^{-1} \begin{bmatrix} \lambda d(\rho+\mu) \\ dw \\ \lambda d(\rho+\mu) \\ dw \\ gd(\rho+\mu) \\ 0 \end{bmatrix} \tag{12}$$

An important difference between H and the standard Hessian of the neoclassical theory is that it is *not symmetric*. Hence we should not be surprised by *asymmetry* of cross-price effects. This has important implications for standard tests of optimality of input demand functions use in industrial organization. On the right-hand side of (12), the terms dw and $\lambda d(p+\mu)$ appear twice. This arises because we are dealing essentially with two (related) firms. For any change in w, the level of employment in both firms will change. It can be easily verified that the cofactors H_{26} and H_{46} in H will not in general be zero. Thus H^{62} and H^{64} in H^{-1} will be non-zero and hence, $d\lambda$ is non-zero, whenever dw is non-zero. The firm's shadow price of capital changes whenever the wage rate changes, and hence, in this model, it is not meaningful to talk about the partial derivatives $\partial K/\partial w$, holding the flow price of capital, $(\rho+\mu)\lambda$, constant. But this is precisely what is invoked in tests of the symmetry of cross-price effects for finished capital and labor. Without recognizing the endogeneity of the price of capital services, the literature equates the price of capital services to the flow price of purchased raw capital. This assumption is false when the firm produces its own capital.

3.1 A Tractable Special Case

It is instructive to investigate a special case of the model. Assume that finished capital, raw investment goods, and labor are used in fixed proportions in the machinery sector. The short-run demands for the factors of production are analyzed using Pontryagin's maximum principle (1962).[7] Define a = output of machines/finished capital; b = labor/finished capital; c = raw investment goods/finished capital.

The long-run equilibrium conditions, (7), (8), and (9) are simple.

The average cost of producing a machine is:

$$\frac{1}{a}(\rho + \mu)\lambda + \frac{b}{a}w + \frac{c}{a}g.$$

In long-run equilibrium, the shadow (demand) price of capital, λ. Hence,

$$\frac{1}{a}(\rho + \mu)\lambda + \frac{b}{a}w + \frac{c}{a}g = \lambda.$$

Equivalently, the shadow price of capital is

$$\lambda = -(bw + cg)/(\rho + \mu - a). \tag{13}$$

The inequality $a > \rho + \mu$ is a necessary condition for the machinery sector to be viable.

[7] See also Arrow and Kurz (1970).

Equation (10) for this specification becomes

$$\begin{aligned}\frac{a}{b}L_2 &= \mu(K_1 + K_2)\\ &= \mu(K_1 + \frac{L_2}{b}).\end{aligned}$$

Equivalently,

$$L_2\frac{(a-\mu)}{b} = \mu k_1. \tag{14}$$

We thus obtain four equations (5), (6), (13), and (14) in four unknowns K_1, L_1, L_2, and λ. The system of equations in (11) reduces to:

$$\begin{bmatrix} PF_{11} & PF_{12} & 0 & -(\rho+\mu) \\ PF_{21} & PF_{22} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{a-\rho-\mu}{b} \\ -\mu & 0 & \dfrac{(a-\mu)}{b} & 0 \end{bmatrix} \begin{bmatrix} dK_1 \\ dL_1 \\ dL_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda d(\rho+\mu) \\ dw \\ dw + \dfrac{c}{d}dg \\ 0 \end{bmatrix}$$

Solving, we get:

$$\begin{bmatrix} dK_1 \\ dL_1 \\ dL_2 \\ d\lambda \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -PF_{22}B(a-\mu)/b & PF_{12}B(a-\mu)/b & -PF_{22}(a-\mu)(\rho+\mu)/b & 0 \\ PF_{21}B(a-\mu)/b & -PF_{11}B(a-\mu)/b & -PF_{21}(a-\mu)(\rho+\mu)/b & 0 \\ -PF_{22}Bb\mu & PF_{12}B\mu & -PF_{22}\mu(\rho+\mu) & -AbB \\ 0 & & -(a-\mu)A/b & 0 \end{bmatrix}\begin{bmatrix} \lambda d(\rho+\mu) \\ dw \\ dw \\ 0 \end{bmatrix} \tag{15}$$

where $\Delta = -(a-\rho-\mu)A\ (a-\mu)$, $A = P^2(F_{11}F_{22} - F_{12}^2)$, $B = (a-\mu-\rho)/b$.

Since F is concave in K_1 and L_1, Δ is negative, since $a > \rho + \mu$.

Several interesting implications follow from the system of equations in (15). Since only the firm's aggregate demand for labor is ever observed, we investigate the change in $L = L_1 + L_2$ for a given change in w. This is given by

$$\frac{\partial L}{\partial w} = \frac{\partial L_1}{\partial w} + \frac{\partial L_2}{\partial w} = \frac{-P(a-\mu)}{\Delta b}\left[\frac{-F_{K_1K_1}(a-\mu-\rho)}{b} + F_{K_1L_1}(\rho+\mu)\right] + \frac{P\mu}{\Delta}\left[F_{K_1K_1}\frac{(a-\mu-\rho)}{b} - F_{L_1L_1}(\rho+\mu)\right]. \quad (16)$$

Rearranging the terms on the right hand side of (16), we obtain

$$\frac{\partial L}{\partial w} = \frac{-P\mu(\rho+\mu)}{\Delta}\left\{\frac{F_{K_1K_1}}{b^2} + \frac{2}{b}F_{K_1}L_1 + F_{L_1L_1}\right\} + \frac{aP}{\Delta}\left\{F_{L_1K_1}(\rho+2\mu) - F_{K_1K_1}\frac{(a-\rho-\mu)}{b}\right\}. \quad (17)$$

The first part of the expression in (17), which does not involve "a," is always negative due to the concavity of F. The sign of the second part is ambiguous. However, if capital is not used in its own production $(a = 0)$ or if capital and labor are complements, in the sense that $F_{K_1L_1} > 0$, then the own price effect is non-positive. Departures from standard neoclassical comparative static results for the law of demand can occur only if $F_{K_1L_1}$ is negative.[8]

[8] Hicks (1939) recognized the possibility of a similar peculiarity. He called this phenomenon a "regression" and remarked that it is hard to reconcile it with common intuition. Our model shows how it might occur.

The effect of the change in w on the equilibrium stock of capital is given by

$$\frac{\partial K}{\partial w} = \frac{Pa}{\Delta b}\Big[F_{K_1L_1}(a-\mu-\rho) - F_{L_1L_1}(\rho+\mu)\Big]. \tag{18}$$

Again, if capital and labor are complementary in production, $\frac{\partial K}{\partial w}$ is negative. Thus, the qualitative properties of the aggregate demand for labor and capital stock are similar to those in the neoclassical theory of the firm.

The signs of individual terms like $\frac{\partial K_1}{\partial w}$ are also ambiguous, if $F_{K_1L_1}$ is negative. Thus

$$\frac{\partial K_1}{\partial w} = \frac{P(a-\mu)}{\Delta b}\left[\frac{F_{K_1L_1}(a-\rho-\mu)}{b} - F_{L_1L_1}(\rho+\mu)\right]. \tag{19}$$

Similarly ambiguity characterizes $\frac{\partial K_2}{\partial w}$, $\frac{\partial L_1}{\partial w}$ and $\frac{\partial L_2}{\partial w}$ if $F_{K_1L_1} < 0$.

Thus, the direction of the change in the aggregate demand for the factors, $\frac{\partial L}{\partial w} = \partial L_1/\partial w + \partial L_2/\partial w$, is uncertain, although a positive change is less plausible. Similar remarks apply to $\frac{\partial K}{\partial w} = \partial K_1/\partial w + \partial K_2/\partial w$.[9] We now turn to the analysis of short-run dynamics.

4 Dynamics

In this section, we characterize the time paths of capital stock and labor when the firm moves from one long-run equilibrium to another. For this—and only for this—purpose, we need a dynamic optimization model. We assume that the firm maximizes the present value of all future net cash flows, as given by the functional,

$$V = \int_{\rho}^{\infty} [PQ - (L_1 + L_2)w] e^{-\rho t} dt. \tag{20}$$

[9] This is similar to the discussion of the Wicksel effect in capital theory.

The constraints are:

$$L_2 = bK_2 = b(K - K_1) \tag{21}$$

$$\dot{K} = aK_2 - \mu K = (K - K_1) - \mu K, \quad K(0) = K_0 \tag{22}$$

$$Q = F(K_1, L_1). \tag{23}$$

The current value Hamiltonian is, making use of (21) and (23),

$$He^{\rho t} = PF(K_1, L_1) - [L_1 + b(K - K_1)]w - c(K = K_1)g + \lambda[a(K - K_1) - \mu K]. \tag{24}$$

The necessary conditions from the Pontryagin Maximum Principle are:[10]

$$H_{K_1} = PF_1 + \rho w + cg - \lambda a = 0 \tag{25}$$

$$H_{L_1} = PF_2 - w = 0 \tag{26}$$

$$\dot{\lambda} = bw + cg + \lambda(\rho + \mu - a) \tag{27}$$

$$\lim_{t\to\infty} \lambda e^{-\rho t} = 0 \tag{28}$$

In long-run equilibrium, $\dot{\lambda}$ and $\dot{K} = 0$, and we obtain

$$bw + cg + \lambda(\rho + \mu - a) = 0.$$

[10]Since the constraining differential equation (22) is not concave in K and K_1, the standard argument (Uzawa, 1964) for establishing the existence and uniqueness of an optimal policy does not apply for our problem. However, Cesari (1965) demonstrates the existence of optimal policy for problems, in which the Hamiltonian is concave in the control variables but not necessarily in state variables. Drandakis and Hu (1968) have extended the above results to problems involving infinite time horizon.

From equation (25), we obtain

$$PF_1 = (\rho + \mu)\lambda. \tag{29}$$

Conditions (22), (26), (27), and (29), with time derivatives equated to zero are exactly the same as (5), (6), (12), and (14), which were used to derive comparative static results.

The differential equations (22) and (27) should be solved to obtain the time paths of K and L. The solution for equation (27) is:

$$\lambda(t) = \Big[D[\exp\Big\{(\rho + \mu - a)t\Big\}] - bw\Big]/(\rho + \mu - a), \tag{30}$$

where D is a constant to be determined from the initial condition K_0. Our assumption that $\rho + \mu < a$ assures us that the transversality condition is satisfied. Using (25) and (26), K_1 can be expressed as a function of $\lambda(t)$. Substituting this expression for K_1 in (22), we see that the differential equation for $\dot{K}$ involves $\lambda(t)$. Making use of (30) for $\lambda(t)$, $\dot{K}$ could be expressed simply as a function of time, D, and the parameters. This is an ordinary differential equation in K, its degree being determined by F. The constant D is determined by obtaining a particular solution of the differential equations passing through K_0. If F is of degree greater than two, analytical solutions are not easy to obtain. Yet, this formulation of the firm's capital acquisition is much simpler than the ones encountered in many optimal control problems. Problems in control theory typically involve a set of nonlinear simultaneous differential equations with split end point conditions.

Necessary conditions (22), (25), and (26) have important econometric implications. Since $\lambda(t)$ is known except for a constant, $\lambda(0)$, the above three equations could be solved, and the variables K_1, L_1, and K could all be expressed as a function of time, the wage rate, and $\lambda(0)$. Since we assume fixed proportions in the capital producing sector, L_2 and K_2 are simply proportional to gross investment. Thus, we get a mathematical relation between K, L, and I if $\lambda(0)$ is a constant. This relation can be used for purposes of estimation of a time series for one firm, except for one difficulty. $\lambda(0)$ will be a constant only if no unanticipated change in any price occurs, or in other words, the target is fixed. Since this is quite unlikely, this relation should be estimated, assuming $\lambda(0)$ to be a time varying parameter that is constant in periods of constant prices and no shocks. However, a very useful relationship that does not suffer from this difficulty is equation (26):

$$PF_2(K_1, L_1) = w.$$

Since K_2 and L_2 are proportional to gross investment, we can rewrite this relation as

$$PF_2(K - K_2, L - L_2, I) = w = G(K, L, I).$$

If, for example, $F(K_1, L_1)$ is Cobb-Douglas, then the following relation is obtained:

$$\rho(K - I/c)^{\alpha}(L - bI/c)^{\rho-1} = w/p.$$

Taking logarithms,

$$\log \rho + \alpha \log[K(1 - I/cK)] + (\rho - 1) \log L(1 - bI/cK) = \log w/p.$$

Simplifying and approximating the logarithms, we get

$$(\rho - 1)\log L + \alpha \log K - (\rho - 1)bI/cL - \alpha I/cK - \log(w/p) + \log \rho = 0.$$

The validity of this equation can now be tested, since this does not depend upon $\lambda(0)$. The implicit assumption, of course, is that the firm at each instant expects the current wage rate to prevail forever . Whether this relation and the model behind it is a useful one is an empirical question.

4.1 The Elasticity of Demand for Labor

We now make some simplifying assumptions to show that the firm's demand for labor can be *more* elastic in the short run than in the long run. Assume that machines, once produced, last forever, i.e., $\mu = 0$. Then the machinery sector will be closed in the long run, and the labor demand L_2 will be zero. If F_{12} is negative and "b" is not large enough, then $\frac{\partial L_1}{\partial w} < 0$, from (16), in the long run. In other words, the "Wicksell Effect"[11] does not dominate, and hence the firm employs less labor in the long run. Let us assume that the firm is in its long-run equilibrium to start with and there is a once and for all increase in the wage rate. The short-run demand for labor in the two sectors depends upon K_1, which in turn depends crucially upon $\lambda(0)$. If K_1 is known, (26) could be used to determine L_1 and (21) to determine L_2. Since there is net addition in the long run to the capital stock, the firm has to produce capital in the short run. The shadow price of capital will then be positive under very general conditions. Equation (25) then suggests that some capital

[11] See Kurz and Salvadori (1997) and also Wicksell (1896), Wicksell (1934).

will be withdrawn from the final product sector. Again, if the machinery sector is highly labor intensive (large "b") and the output-capital ratio (a) is small, the capital withdrawn from the first sector will be large. Because of this and since F_{12} is negative, the demand for labor in the first sector could increase in the short run, in spite of an increase in w.

When $K - K_1$ is positive, the short-run demand for labor in the machinery sector is positive. Thus, the firm will hire (labor) to produce machines and product in the short run. Once the desired machines are all produced, the firm will reduce its total demand for labor to a lower level than the initial one. Thus, the firm's aggregate demand for labor could be much more elastic in the short run than in the long run.[12] This peculiarity arises only because the firm is producing its own capital stock. This is the basis, of Treadway's theorems for the short-run behavior of the firm. The appendix examines his model in detail and exposits its special features.

5 A Quadratic Example

To fix ideas, it is useful to consider a quadratic case. As we show in the appendix, Treadway's approximation analysis is exact when the technology is quadratic. Using this model, we can show how efficient lagged adjustment can rationalize static inefficiency as an autoregressive process. Ahn and Sickles (2000) assume that static inefficiency obeys an autoregression. We show that their econometric model can be rationalized by a dynamically efficient model of adjustment costs.

[12]A mathematical demonstration of this possibility is presented in the appendix.

Let the production function for the final product be

$$F(K_1, L_1) = \alpha K_1 + \beta L_1 - (\gamma/2)K_1^2 + \zeta K_1 L_1 - (\epsilon/2)L_1^2. \tag{31}$$

Then,

$$F_{K_1} = \alpha - \gamma K_1 + \zeta L_1 = \lambda a - bw - cg \tag{32}$$

$$\text{and} \qquad F_{L_1} = \alpha - \epsilon L_1 + \zeta L_1 = w. \tag{33}$$

Equation (33) implies that $L_1 = (\zeta K_1 + \beta - w)/\epsilon$.

Substituting for L_1 in(32), we obtain

$$\alpha - \gamma K_1 + (\zeta/\epsilon)(\zeta K_1 + \beta - w) = \lambda a - bw - cg,$$

$$\text{so} \qquad K_1 = [\lambda a - bw - cg - \alpha - (\zeta/\epsilon)(\rho - w)]/[(\zeta^2/\epsilon) - \gamma]. \tag{34}$$

From (22), we have

$$\dot{K} = a(K - K_1) - \mu K = (a - \mu)K - aK_1. \tag{35}$$

Substituting for K_1 from (34), the above equation becomes:

$$\dot{K} = (a - \mu)K - a[\lambda a - bw - cg - \alpha - (\zeta/\epsilon)(\rho - w)]/[(\zeta^2/\epsilon) - \gamma]. \tag{36}$$

Differentiating (36) with respect to t, we have

$$\ddot{K} = (a - \mu)\dot{K} - a^2\epsilon\dot{\lambda}/(\zeta^2 - \gamma\epsilon).$$

Substituting for $\dot{\lambda}$ from (27), we obtain

$$\ddot{K} = (a - \mu)\dot{K} - a^2\epsilon[bw + cg + \lambda(\rho + \mu - a)]/(\zeta^2 - \gamma\epsilon). \tag{37}$$

Substituting for λ, we obtain

$$\ddot{K} = \rho\dot{K} - (a - \mu)(\rho + \mu - a)K - C, \tag{38}$$

where $C = a\epsilon(\rho + \mu - a)[bw + cg + \alpha + (\zeta/\epsilon)(\rho - w)]/(\zeta^2 - \gamma\epsilon)$.

The long-run equilibrium stock of capital is obtained by setting all derivatives to zero. Thus,

$$K^* = c/(a - \mu)(a - \rho - \mu).$$

The characteristic roots of the differential equation (37) are real and are of opposite signs. Ignoring the positive roots that produce solutions violating transversality conditions, the solution to (38) is

$$K(t) = K^* + (K_0 - K^*)e^{\theta t}, \tag{39}$$

where $\theta = \left[P - \left\{P^2 - 4(a - \mu)(\rho + \mu - a)\right\}^{\frac{1}{2}}\right]/2$, which is < 0.

Differentiating (39) with respect to t,

$$\dot{K} = \theta(K_0 - K^*). \tag{40}$$

This is a version of the flexible accelerator model of Treadway (1971).

Equating (35) and (40), we have, in the short run,

$$K_1 = [\theta K^* + (a - \mu - \theta)K_0]/a. \tag{41}$$

$$L_1 = \Big\{\delta/a[\theta K^* + (a - \mu - \theta)K_0] + \rho_{-w}\Big\}/\epsilon. \tag{42}$$

We also have

$$L_2 = b(K_0 - K_1) = b\Big[\frac{\mu + \theta_K}{a} - \frac{\theta K^*}{a}\Big]. \tag{43}$$

The long-run equilibrium value of L_1 is obtained by substituting K_1^* in (33) and solving for L_1. Hence,

$$L_1^* = [\delta K^*(1 - \mu/a) + \rho - w]/\epsilon. \tag{44}$$

Also,

$$L_2^* = b_\mu K^*/a. \tag{45}$$

The question now is whether $L_1 + L_2$ could be greater than $L_1^* + L_2^*$. The difference between the short-run and long-run demand for labor is

$$L_1 + L_2 - L_1^* - L_2^* = (1/a\epsilon)[K^* - K_0][-b\epsilon\theta + \delta(\theta - a) + \mu\delta - \mu b\epsilon]. \tag{46}$$

When the firm is not in equilibrium at time 0, $K^* - K_0$ is not zero. Assume that it is positive.

In the second square bracket in (46), the first term is always positive and the last term is always positive. The remaining terms could be of either sign. Thus, the ambiguity in the ranking of the short-run and long-run response is apparent. It is thus possible for the labor demand to be greater in the short run than in the long run. For example, consider the case when $\mu = 0$ and $\delta \leq 0$.

Note further that due to adjustment costs, applying a static model for long-run equilibrium as a consequence of such adjustment costs, even though the firm is dynamically efficient. Silva et al. (2020) and Tsionas et al. (2020) consider estimation of versions of an adjustment cost model without firm-specific capital. Our model rationalizes the first order autoregressive model for static inefficiency developed and estimated in Ahn and Sickles (2000). Our derived adjustment model is in autoregressive form. It is characterized by dynamic efficiency, albeit static inefficiency.

6 Summary and Conclusion

This paper discussed the theory of the firm under adjustment costs for quasifixed, firm-specific factors. Treadway (1970) is the inspiration for this literature. He makes some counterintuitive claims about how introducing adjustment costs challenges the neoclassical theory of the firm. His work is the basis for subsequent work on productivity and efficiency measured approaches. McLaren and Cooper (1980) derive a version of Hotelling's Lemma for Treadway's model, but without firm-specific capital.

We clarify claims in Treadway's paper. His results rely on aggregating two sectors of the firm: (a) the sector producing capital in place; and (b) the sector using installed capital to produce final output. Disaggregating the firm as we have, standard neoclassical results hold for each sector but not for the firm aggregate. We consider the case that firm specific capital cannot be directly purchased in the capital goods market although raw investments goods can be. Aggregate labor demand schedules can be upward sloping in terms of wages under conditions we derive. Disaggregated within-sector demands obey the usual law of demand. In the presence of adjustment costs, static inefficiency, as estimated by frontier methods can arise because of mismeasured adjustment costs in a dynamically efficient model.

Appendix

A Closer Look at Treadway (1970)

We examine Treadway's (1970) analysis in greater detail and show that his approximate analysis is correct only in the globally quadratic case. We first examine the proofs of Treadway' s theorems for the short run. It is well known that in any dynamic optimization problem, the optimal path is a saddle path. Being aware of this, Treadway linearly approximates the differential equation system in the neighborhood of the saddle point. He then discards the positive characteristic roots to satisfy transversality conditions. The resulting adjustment model will be exactly the same as the "flexible accelerator" model, implying maximum adjustment in the first instant.

Treadway's theorems for the short run are correct only if production is globally quadratic; otherwise, the results may approximate the true short-run results.[13] A careful analysis of the problem of approximating solutions around a saddle point has been made by Mirrlees (1967). His important conclusion is that the relative sensitivity of the solution is very high when, for example, the capital-labor ratio is a variable in his model. He also points out that unless the capital stock, for example, increases all along the optimal path from the initial position to the long-run level, the approximating capital stock path may not have the same time derivative as the true one.

Treadway attributes the perverse relation between the short-run and long-run coefficients again to the explicit introduction of time. He states that the static theory, based on the Le Chatelier principle (Samuelson, 1947), implies that the short-run effect is smaller

[13] In his equation (14), Treadway drops $L(t)$ altogether; this appears to be possible only if F is assumed to be linearly homogenous in K and L.

than the long-run effect. The relevant question is whether in his model an extended Le Chatelier principle leads to the same implications as the dynamic optimization theory.

Samuelson (1947), in deriving the principle, assumes that capital cannot be changed at any cost in the short run. Now assume that the short-run supply price of a unit of capital varies directly with the level of capital stock bought and inversely with the level of employment. Let static equilibrium conditions be

$$PF_K = (\rho + \mu)\lambda.$$
$$PF_L = w.$$

We assume that the price of capital is

$$\lambda = \lambda_0 + \rho(t)K + \gamma(t)L,$$

where "t" is the length of time after a disequilibrium occurred, say, an increase in wage rate. When $\rho(t) \to \infty$, for any fixed t, capital is completely fixed. This case corresponds to the Marshallian short-run analysis. Samuelson developed the Le Chatelier principle in this context. If $\rho(t) = 0$, then capital is freely variable. Positive values of ρ correspond to the Eisner and Strotz analysis of the short run.

Differentiating the above equilibrium conditions totally, for fixed t, we obtain

$$\begin{bmatrix} PF_{KK} - \rho(t) & PF_{KL} - \gamma(t) \\ PF_{LK} & PF_{LL} \end{bmatrix} \begin{bmatrix} dK \\ dL \end{bmatrix} = \begin{bmatrix} (\rho + \mu)d\lambda_0 \\ dw \end{bmatrix}.$$

Solving the above system,

$$\begin{bmatrix} dK \\ dL \end{bmatrix} = \begin{bmatrix} & PF_{LL} & -PF_{LK} \\ 1/\Delta & & \\ & -[PF_{KL} - \gamma(t)] & [PF_{KK} - \rho(t)] \end{bmatrix} \begin{bmatrix} (\rho + \mu) d\lambda_0 \\ dw \end{bmatrix},$$

where $\Delta = [PF_{KK} - \rho(t)][PF_{LL} - \gamma(t)] - [PF_{KL} - \gamma(t)]PF_{KK}$. It is apparent that there is no symmetry of the cross-price effects as long as $\gamma(t)$ continues to be non-zero, whether it is the short run or long run.

We next examine various partial derivatives to determine the relation between the short-run and long-run own price effects. Assume $\gamma(t) = 0$. The short run effect in the sense of Eisner and Strotz is, for fixed t, given by,

$$\begin{aligned} ES = \frac{\partial L}{\partial w} &= [PF_{KK} - \rho(t)] / \Big[[P_{KK} - \rho(t)]PF_{LL} - P^2 F_{LK} \Big] \\ &= 1/[PF_{LL} - \Big\{ P^2 F_{LK}/PF_{KK} - \rho(t) \Big\}]. \end{aligned}$$

The Marshalian short-run effect is obtained letting $\rho(t)$ tend to infinity in the above equation. It is easily seen to be

$$M = \frac{\partial L}{\partial w} = 1/PF_{LL}.$$

The long-run effect is obtained by letting $\rho(t) = 0$:

$$LR = \frac{\partial L}{\partial w} = F_{KK}/P[F_{KK}F_{LL} - F_{KL}^2].$$

Since F is concave in K and L, then the following holds:

$$M \quad < \quad ES \quad < \quad LR.$$

If, for example, $\rho(t)$ does not tend to zero as $t \to \infty$ or if $\gamma(t)$ is non-zero, such rankings are again not possible. Our present analysis can be criticized for the *ad hoc* nature of the assumptions about $\rho(t)$. We take it as a point of departure.

From the particular behavioral assumptions we make, we could easily deduce their long-term implications, i.e., whether $\rho(t)$ and $\gamma(t)$ tend to zero as $t \to \infty$ or not. From this we could find out the applicability of the static model. If, for example, $\rho(t)$ tends to a positive constant, then a static model which assumes an upward sloping supply curve of capital is the relevant one. The exact time shapes of $\rho(t)$ and $\gamma(t)$ are not derived in such an analysis. Doing so would require solving simultaneous nonlinear differential equations with split end point conditions to obtain the precise path, a task we leave for another occasion.

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