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Allocating Scarce Information

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ABSTRACT

Sender conveys scarce information to a number of receivers to maximize the sum of receiver payoffs. Each receiver's payoff depends on the state of the world and an action she takes. The optimal action is state contingent. Under mild regularity conditions, we show that the payoff of each receiver is convex in the amount of information she receives. Thus, it is optimal for Sender to target information to a single receiver. We then study four extensions in which interior information allocations are optimal.

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1 Introduction

Economics is often defined as the study of "allocating *scarce* resources." When those resources are goods or services, economists have achieved a largely complete understanding of the problem. Under the conditions of the First Welfare Theorem a competitive general equilibrium yields a Pareto optimal allocation of resources. When those conditions do not hold–perhaps due to the presence of externalities–governments have a role to play. Importantly, for our purposes here, a utilitarian social welfare function leads the social planner to an interior solution where resources are shared among agents.

It is tempting to think that the allocation of information behaves similarly to that of goods and services. The objective of this paper is to analyze a general setting where a benevolent planner allocates scarce information that is then used by receivers to tailor their actions. We show that, even with a utilitarian social welfare function, the general information allocation problem behaves rather differently to the allocation of goods and services. In particular, individual payoffs are convex in information and so an interior solution may not be optimal.

We analyze a model in which Sender decides on how to allocate information about an unknown state of the world to each of a finite number of receivers. Receivers each have their own payoff-relevant state of nature, independently drawn from a common distribution, and take an action based on their posterior belief about the state. Each receiver's payoff depends on their state and action taken. Sender chooses an allocation of information to maximize the sum of receiver payoffs.

It is easy to see that a number of practical information-provision problems fall into this class. The provision of information about crime rates in various communities within a city is one. The provision of public health information during a crisis (such as HIV or COVID-19) is another. In general, an important role of government is to provide information to the public, and this typically involves information being scarce.

The novel ingredient in our model is that the information available to Sender is scarce. We capture scarcity in two ways. First, Sender is limited to choosing an allocation (weights), one for each receiver, the sum of which must be less than or equal one. A receiver's allocation determines their signal, which comprises of a blend of two signal structures: one informative (Blackwell, 1953), and one completely uninformative. The greater the receiver's allocation, the more weight is placed on the informative signal in this blend, proportionally increasing the quality of information allocated to the receiver. Second, Sender can only draw a single message from each receiver's signal, and report the resulting vector of messages (a "public report") which is observed by all receivers.

A key insight is that a receiver's payoff is convex in the quality of their information. This follows from two sources of benefits to a receiver from increasing their allocation. First, fix a receiver's current actions, which already cater towards responding to any state(s) of which a report is suggestive. Increasing the quality of information increases the frequency of observing a report on such states. In turn, this increases the likelihood that the receiver's current actions are taken "correctly" to their benefit. Second, the receiver also benefits by further adjusting their actions to suit the higher quality information obtained. Continuing to increase the receiver's allocation induces further adjustments in their actions, in turn increasing the magnitude of benefits from both sources. Combined, these imply that the receiver experiences increasing returns in their allocation of information.

We illustrate this core intuition with a simple numerical example. Consider a Government (Gov) trying to provide advice about the severity of the COVID-19 pandemic to two symmetric (groups of) citizens. The state of severity for citizen i can either be 0 ("low") or 1 ("high"). States are equally likely and independent across citizens. Each citizen chooses a response $a_i \in [0, 1]$. If the state is θ_i when a_i is chosen, citizen i's payoff is given by the quadratic loss form $-(\theta_i - a_i)^2 + \frac{1}{4}$.

Gov is limited in the quality of advice she can provide to citizens. She is constrained to a single report, and chooses how to divide it among each citizen. Gov first chooses $x \in [0, 1]$ such that $x_1 = x$ and $x_2 = 1 - x$ is the proportion of the report dedicated to citizen 1's and 2's advice. Citizens observe a single report $m = (m_1, m_2)$ prior to their choice of response. The component of the report relevant to citizen i's state, m_i, can take on either l or h, and is proportionally more likely to match the underlying state from an increase in the receiver's allocation. That is, $Pr(m_i = h|\theta_i = 1) = Pr(m_i = l|\theta_i = 0) = \frac{x_i}{4} + \frac{1}{2}$, which captures the informativenss of Gov's report to citizen i, is increasing in x_i.

Let us compute each citizen's payoff. First, let $q = Pr(\theta_i = 1)$ be the posterior belief formed on the state being severe. Fixing belief q, any citizen's optimal response is $a_i = q$. This yields him a (conditional) expected payoff of $u(q) = q^2 - q + \frac{1}{4}$, which is convex in q. Second, as citizens' actions and states are independent, a citizen only cares about their message and the proportion of the report dedicated to him, m_i and x_i. Hence, their belief $Pr(\theta_i = 1|x_i, m_i)$ is given by $q_h(x_i) = \frac{x_i}{4} + \frac{1}{2}$ and $q_l(x_i) = \frac{-x_i}{4} + \frac{1}{2}$. Both are *linear* in x_i. From here, the citizens' payoff is given by $\frac{1}{2}u(q_l(x_i)) + \frac{1}{2}u(q_h(x_i)) = \frac{x_i^2}{16}$, i.e., the product of expected payoffs given a belief upon observing Gov's report, and the probability of observing a report. This confirms our key insight: a citizen's payoff is *convex* in their allocation.

A direct consequence of this is that Sender optimal allocates all information to a single receiver. That is, in our baseline model it is optimal for Sender to *fully target* information to one receiver. We go on to consider elaborations of the baseline model that admit interior or even no allocation of information. Four relevant environments are emphasized in the paper.

First, we consider more general information production functions. Convex production functions lead to full-targeting, while with sufficiently concave production functions, interior allocations can be optimal.

Second, we consider non-utilitarian welfare functions. We first consider when Sender cares about a weighted sum of efficiency (sum) and equality (min). Due to Receivers' increasing returns to information, we find that Sender switches from one extreme of fully-targeting, to the other extreme of a perfectly Rawlsian allocation, when her concern about equality is sufficiently weighted. Since Sender's allocation directly affects the spread over actions taken by receivers, we also consider circumstances where Sender may have an additional persuasion incentive to maximize or minimize this spread. We demonstrate that an increase in the preference for the former leads Sender towards a more targeted allocation, while an increase in preference for the latter does the opposite, inducing a less targeted or even no allocation.

Third, we consider a setting where receivers strategically interact. An increase in the importance of strategic considerations induces a receiver towards taking ”informationally optimal” actions, i.e., actions which best utilize the information available to him, to “strategically optimal actions”, i.e., actions taken considering how other receivers respond. Strategic concerns weaken Sender's incentive to target when (i) a receiver's strategic concerns directly conflict with their use of information, which weakens increasing returns in their *own* allocation, and (ii) a receiver's payoff is concave enough in *other* receivers' actions, such that the receiver benefits from other receivers taking less spread out actions. We investigate several environments which satisfy both conditions, a notably one being when receivers prefer to conform to other receivers' actions and losses from failing to conform are sufficiently convex. In these environments, we show that an increase in the salience of strategic concerns induces a less targeted or even no allocation from Sender.

Fourth we investigate a setting where the quality of information available to Sender is unknown to receivers. Two additional insights arise. Senders of different types (in terms of the quality of their information) disagree over their preferred (perfect Bayesian) equilibria. This is different from complete information, where all receivers prefer targeting a single receiver. This is because targeted receivers tend to “overreact” based on the expectation of the quality of information they are allocated, relative to what a Sender actually has. Senders with sufficiently low quality of information opt out entirely, i.e., choose not to disclose anything, in some equilibria, fearing “overreacting” receivers. This is never the case with complete information, where receivers always update correctly and act accordingly, so all Senders provide information.

Our paper relates to the now considerable literature on “Bayesian Persuasion” pioneered by Kamenica and Gentzkow (2011) and Rayo and Segal (2010).[1] Unlike

[1]Bayesian persuasion is a part of the larger information design (Bergemann and Morris, 2019) literature. Antecedents include Aumann and Maschler (1995), Brocas and Carrillo (2007), and

in Bayesian Persuasion models, in our environment Sender and receiver's preferences are completely aligned. Furthermore, Sender does not control the type of information, but rather its quality. From a technical perspective, this changes our mode on analysis from linear programming (Kolotilin, 2018, Dworczak and Martini, 2019) to majorization and monotone comparative statics (Milgrom and Shannon, 1994).

Our paper also connects to a series of papers that investigate exogenously constrained information design. Tsakas and Tsakas (2021) and Le Treust and Tomala (2019) investigate the role of exogenous noise in persuasion. Ichihashi (2019) studies whether constraining Sender's information can improve efficiency. Another body of work studies the setting where communication is coarse in the sense that the cardinality of the message space is restricted. Dughmi et al. (2016), Le Treust and Tomala (2019) and Aybas and Turkel (2019) are notable examples.[2]

Our paper complements this line of work in two ways. First, we focus on an *allocation problem*, rather than the *persuasion problem*. This focus motivates our allocation-like budget constraint, which results in our allocation problem speaking to the issue of "to whom should Sender disclose information." Second, we analyze the implications of varying Sender's alignment in incentives with that of receivers, and how it connects to Sender's optimal allocation. In particular, we also discuss how strategic incentives *between* receivers play a role in the Sender's allocation of information, whereas existing work focuses on the single-receiver problem.

There is also a strand of literature focusing on costly information provision. Gentzkow and Kamenica (2014) study a persuasion problem where there is a cost to Sender proportional to the informativeness of the signal chosen. Matyskova and Montes (2021) consider a persuasion problem where Receiver can choose to acquire information at a cost after observing Sender's disclosure of information. Our model can be interpreted as one where the production of information itself is

Friedman and Holden (2008).

[2]Rather than implementing explicit constraints, several papers instead study when simple information structures are sufficient to achieve an optimal payoff for Sender, examples including Kolotilin et al. (2021) and Kolotilin (2018), Kolotilin and Wolitzky (2020)).

costly for Sender, and Sender is restricted to a finite budget and a specific information production function.[3] In general, our paper complements this literature by focusing on how information is allocated in the absence of preference misaligned between Sender and Receiver(s), rather than than the (important) strategic question of "how to persuade a Bayesian" that originally motivated the now extensive Bayesian persuasion literature.

The paper proceeds as follows. Section 2 introduces the main framework. In Section 3 we provide general results. Sections 4 considers four relevant extensions to the main framework. Section 8 contains some brief concluding remarks. All proofs and further technical details are relegated to Appendices A and B.

2 The Model

In this section, we introduce the basic framework underpinning the main results of this paper. The key mathematical model set-up is introduced in Section 2.1 and 2.2, while a discussion of assumptions / rationale is provided in Section 2.3.

2.1 Statement of the Problem

Model: There are two types of players: a Sender and $N \geq 2$ number of symmetric receivers indexed by $i \in \mathcal{N} \equiv \{1, 2, ..., N\}$. For each receiver, there is a state space $\Theta_i = \Theta$, which is a compact subset of $[0, 1]$ and satisfies $|\Theta| > 1$. States are independently and identically distributed by $F \in \Delta\Theta$ (where ΔS denotes the set of probability distributions over set S). Each receiver has an symmetric action space A, which we assume to be a compact metric space. When action a is taken

[3]A number of other papers have imposed "soft" costs on the Sender by explicitly limiting her commitment power. These are mainly applicable to scenarios where Sender and receivers' incentives are misaligned, which differs from our primary focus. For instance, Lipnowski et al. (2021) and Min (2021) analyze a persuasion problem where the Sender can manipulate the message observed under an experiment with positive probability. Nguyen and Tan (2021) consider a persuasion problem where Sender commits to an information structure, privately observes the signal realization, then sends a message to receiver whose cost is proportional to its proximity from the true observed signal realization.

by (representative) receiver i on state θ, receiver i's payoff is given by $u(\theta, a)$ which we assume to be continuous in (θ, a). We further assume that Sender is utilitarian: given a collection of states $(\theta_1, ..., \theta_N)$ and actions $(a_1, ..., a_N)$ of each receiver, Sender's payoff is $v(\theta_1, ..., \theta_N; a_1, ..., a_N) = \sum_{i=1}^{N} u(\theta_i, a_i)$.

Sender is endowed with information about each receiver's state. More precisely, Sender has access to a collection of signals $\{(G_i, M_i)\}_{i=1}^{N}$. Each signal comprises of a message space $M_i = [0, 1]$ and a mapping $G_i : \Theta_i \to \Delta[0, 1]$. We assume that signals are symmetric across receivers, i.e., $G_1 = \cdots = G_N = G$ and that the conditional density $g(m|\theta)$ exists for all $\theta \in \Theta$. Let $g(m) = \int_0^1 g(m|\theta)dF(\theta)$.

Due to scarcity constraints, Sender is restricted in the precision of information communicated to each receiver. More precisely, Sender first chooses $x \in X = \{x' \in [0, 1] : \sum_{i=1}^{N} x_i \leq 1\}$. We call x Sender's *allocation*, and X her *budget constraint*. We further note that it will often be useful to distinguish between allocations in X, and those under which the budget constraint binds. We denote this set as $\overline{X} = \{x' \in X : \sum_{i=1}^{N} x_i = 1\}$. Given x, receivers observe a vector of realizations $(m_1, ..., m_N) \in M^N$. Each message m_i is independently drawn from the composite signal (G^{x_i}, M), where G^{x_i} is defined by

$$G^{x_i} = x_i G_i + (1 - x_i)\underline{G}_i \tag{1}$$

for $x_i \in [0, 1]$, and $\underline{G}_i$ is completely uninformative about receiver i's state such that for each $\theta_i \in [0, 1]$, $\underline{G}_i$ has conditional density $\underline{g}_i(m_i|\theta_i) = g(m_i)$.[4] Combined, these assumptions imply that an increase in x_i increases the informativeness of the composite signal G^{x_i} to receiver i.

Timing: The timing of the game is as follows

- **Stage 1:** Nature draws $\theta_i \in [0, 1]$ independently for each receiver according to F
- **Stage 2:** Sender chooses $x \in X$.

[4]Observe that for any $\theta_i \in \Theta$, the conditional density of G^{x_i} is given by $g^{x_i}(m_i|\theta_i) = x_i g(m_i|\theta_i) + (1 - x_i)g(m_i)$.

- **Stage 3:** Given $(\theta_1, ..., \theta_N)$, a vector of realizations $m = (m_1, ..., m_N)$ is drawn from M^N, where each $m_i \in M$ is drawn from G^{x_i}. Receivers observe m and Sender's choice of x, update their beliefs about their true state, and simultaneously and independently choose an action $a_i \in A$.

- **Stage 4:** Players obtain their payoffs.

Solution concept: We assume that information about the primitives is common knowledge among all players. Our solution concept can be described as follows. First, under any vector of messages observed, a receiver chooses the action which maximizes their conditional expected payoff. Since states and actions are independent, a receiver only cares about the information content of her message m_i, which is solely dependant on x_i. Aggregating over all messages, one finds that for any $x \in X$, receiver i's payoff is only dependent on x_i and is given by

$$U(x_i) = \int_0^1 \left(\max_{a \in A} \int_{\text{supp}(F)} u(\theta, a)(x_i[g(m|\theta) - g(m)] + g(m))dF(\theta) \right) dm \tag{2}$$

Where we shall assume that $U(1) > U(0)$, i.e., that Receivers strictly benefit from being fully targeted. Furthermore, let $a^*_{m_i}(x_i)$ denote a receiver's optimal action given allocation x_i and message m_i observed drawn from her signal.[5]

Second, Sender chooses the allocation which maximizes her expected payoff, conditional on receivers acting optimally. Labelling Sender's payoff as $V(x) = \sum_{i=1}^N U(x_i)$, we require that Sender chooses[6]

$$x^* \in \arg\max_{x \in X} V(x) \tag{3}$$

We call any such x^* an *optimal allocation*.

[5] If the receiver has multiple optimal actions, let $a^*_{m_i}(x_i)$ denote any one of them.

[6] By continuity of $u(\theta, a)$ in a for all θ, $U(x_i)$ is continuous in x_i. That X is compact then implies that an optimal allocation exists.

2.2 Preliminaries

We will focus heavily on discussing how the degree of targeting by Sender changes with respect to changes in the primitives. Doing so requires a way of defining "more targeted" or "less targeted" allocations. We provide a brief introduction here which is sufficient for our analysis. Further technical details are relegated to Appendix A2.

Majorization For any $x \in X$, let $[x]$ denote the allocation x but with coordinates rearranged in decreasing order, i.e., $[x]_1 \geq [x]_2 \geq \cdots \geq [x]_n$. Take any $x, y \in \mathbb{R}^N$. We say that x *weakly majorizes* y, written as $x \succ_w^M y$, if for all $n = 1, .., N$, $\sum_{i=1}^n [x]_i \geq \sum_{i=1}^n [y]_i$. If $x \succ_w^M y$ and $\sum_{i=1}^N [x]_i = \sum_{i=1}^N [y]_i$, then we say that x *majorizes* y, and write $x \succ^M y$.

In our context, majorization provides a precise criterion for when one allocation is more targeted, i.e., less dispersed, than another. For instance, take any $x, y \in \overline{X}$ such that $x \succ^M y$ holds. Then, both allocations fully exhaust the allocation budget available to Sender, but allocation x is more targeted at a select few receivers / more spread out than allocation y. Weak majorization captures the same concept, but relaxes the requirement of $\sum_{i=1}^N x_i = \sum_{i=1}^N y_i$. That is, one may have either x or y utilise a smaller portion of the allocation budget than the other.

As they will appear through this paper, we also define the "most" and "least" targeted allocations in X as follows.

Definition 1. *We call an allocation $x \in X$:*

- ***Fully targeted*** *if $x_i = 1$ for some $i = 1, .., 0$, and $x_j = 0$ for $j \neq i$.*
- ***Evenly spread*** *if $x_i = 1/N$ for $1 \leq i \leq N$*
- ***No disclosure*** *if $x_i = 0$ for all $i = 1, ..., N$.*

Intuitively, fully targeted allocations are the most targeted allocations available to Sender, leaving a single receiver maximally informed (given G), while all receivers remain completely uninformed. Meanwhile, the evenly spread allocation is the least targeted allocation available to Sender, provided that Sender fully exhausts the allocation constraint $\sum_{i=1}^{N} x_i = 1$. Finally, no-disclosure is both the least dispersed and least informative allocation available to Sender. Such an allocation leaves all receivers completely uninformed.[7]

Non-decreasing and non-increasing Often, we will be discussing how the set of optimal allocations are becoming more-targeted or less-targeted from a change in some underlying parameter. Consider a subset $X(\omega) \subset \overline{X}$ (or X) parameterized by $\omega \in \Omega \subset \mathbb{R}$. We say that $X(\omega)$ is *non-decreasing* (*non-increasing*) with respect to $\succ^M$ in ω if for all $\omega, \omega' \in \Omega$ where $\omega > \omega'$, for any $x \in X(\omega)$ and $x' \in X(\omega')$ which are comparable under the majorization order, i.e., either $x \succ^M x'$ or $x' \succ^M x$ holds, we have $\max\{[x], [x']\} \in X(\omega)$ and $\min\{[x], [x']\} \in X(\omega')$ ($\min\{[x], [x']\} \in X(\omega)$ and $\max\{[x], [x']\} \in X(\omega')$). A similar definition may be applied, replacing $\succ^M$ with $\succ^M_w$, to define $X(\omega)$ being *non-decreasing* (*non-increasing*) with respect to $\succ^M_w$ in ω. Interpretively, $X(\omega)$ is non-decreasing (non-increasing) with respect to $\succ^M$ in ω whenever the collection of allocations cannot become strictly less (more) targeted (where our definition of targeting is that of majorization or weak majorization) from an increase in ω.

2.3 Discussion of assumptions

Symmetry In any scarce allocation problem, there are two factors which drive a social planner's allocation. There are those inherent to the allocation problem itself, and those which arise due to asymmetry in the primitives. Our focus is on a discussion on the former, and so we assume an (ex-ante) symmetric environment to disentangle the core mechanisms driving our results, from those arising from asymmetry. From a technical perspective, symmetry also simplifies the analysis

[7] More precisely, a fully targeted allocation majorizes all elements of $\overline{X}$ and weakly majorizes all elements of X, the evenly spread allocation is majorized by all elements of $\overline{X}$, and no-disclosure is weakly majorized by all elements of X.

by allowing us to utilize powerful results from the majorization and monotone comparative statics literature.

Scarcity Our core assumption is that Sender is restricted by the quality of information available to be supplied to receivers. Furthermore, the restriction operates via considering convex compositions of signals available to Sender subject to the constraint $x \in X$.

Among others, there are two straightforward ways in which this may be interpreted. First, Sender is limited in the amount of time she has to convey useful information that she is endowed with via her report. For example, a government official may have a restricted time slot to convey crucial information about nation wide crime-rates or disease infection rates to citizens. Such constraints may also arise due to citizens possessing limited time and capability to watch news reports. If so, then x_i can be interpreted as the proportion of the report dedicated towards conveying quality information relevant to receiver i's needs. Assuming the time spent discussing receiver i's issue is proportional to the informativeness of Sender's report, one has the informativeness of receiver i's signal G^{x_i} increasing in x_i.

Our problem can be interpreted as an information production problem. Here, x is Sender choice of how to allocate resources towards gathering, analysing and producing certain types of information. Under this interpretation, scarcity now arises due to capacity constraints. For instance, a (monopolistic) news outlet trying to obtain news-worthy content might be restricted by the finite number of reporters available, while a research team gathering data may be limited by its research budget.

We also assume that Sender is restricted in the type of information that she can provide. That is, Sender possesses a single information source G_i per receiver. This is as we aim to study a scarce allocation problem, i.e., limitations of *how much* to provide, rather than a persuasion problem, i.e., limitations on *what* to provide.

Finally, we have assumed that Sender's choice set involves linear compositions of signals. To allow for a non-constant rate of return to informativeness from increasing a receiver's allocation, other more general mixtures are considered in Section 4.1. We also assume that the quality of information is known to receivers. We relax this assumption in Section 4.4.

Use of information In the basic model, we assume that a receiver's payoff is only affected by her own state and action. A natural interpretation for this assumption is that receivers are physically dispersed, i.e., geographically located in different neighbourhoods, such that any one receiver's action and local state has no (material) impact on the other receivers. Of course, one may want to consider the case under which receivers take into account how others will act, in addition to information about their own state, when deciding on what actions to take. We discuss such cases in Section 4.3.

Utilitarian Sender We assume that Sender aims to maximize efficient use of allocated information. Naturally, such an assumption applies to the case where Sender is a social planner, and possesses a welfare function linear in decision makers' utilities.

Alternatively, the assumption captures the scenario where Sender is an "information monopolist", deciding how to produce/package information to be sold to consumers. For example, consider a monopolist news outlet deciding on how to allocate resources towards obtaining relevant local news for each of $i = 1, ..., N$ neighbourhoods. Each neighbourhood contains a unit mass of readers who only care about information pertaining to their own neighbourhood. Readers obtain a payoff of $U(x_i) - p_i$ from purchasing Sender's information, where $U(0) = 0$, and do so if and only if they obtain non-negative utility. If so, then it is immediate that Sender charges $p_i = U(x_i)$ and obtains a payoff of $\sum_{i=1}^{N} U(x_i)$. Hence, the monopolist's problem maps to the problem we study here.[8]

Of course, Sender's preferences may not be perfectly aligned with that of re-

[8]Alternatively, our problem also captures the case where there is a unit mass of receivers who have payoffs (and relevant actions) separable in each "issue" θ_i, i.e., who choose $\boldsymbol{a} \in [0,1]^N$ with utility $u(\boldsymbol{\theta}, \boldsymbol{a}) = \sum_{i=1}^{N} u(\theta_i, a_i)$.

ceivers. Two such cases, i.e., preferences over ex-ante equality and preferences over actions taken by receivers, in Section 4.2.

3 Main result

In this section, we discuss several core insights of the basic framework introduced in Section 2. This lay the groundwork for various extensions explored in Section 4.

We begin our analysis by discussing how a receiver's payoffs vary with respect to her net allocation, x_i. Proposition 1 conveys the key insight of our paper.

Proposition 1. *$U(x_i)$ is convex and increasing in x_i.*

Proposition 1 states that a receiver's payoff displays increasing returns with respect to the allocation x_i. To understand this, we decompose the receiver's benefit from information into two components. First, there is the *accuracy benefit* from increasing the precision of information while fixing the receiver's action. Intuitively, fixing the receiver's current allocation, any action taken under a message is "tuned" towards the direction in which the receiver's beliefs move from increasing the precision of information. Thus, increasing x_i while fixing the receiver's actions, which further moves the receiver's beliefs in this direction, cannot leave the receiver worse off in expectation. Second, there is the *adjustment benefit*. The receiver adjusts their actions (to their benefit) to capitalize on more precise information.

Continuing to raise the receiver's allocation raises both accuracy and adjustment benefits. In the former case, further increases in the allocation induces the receiver to further tune their actions (in the way described above) such as to benefit from further increases in the precision of information. In the latter case, further increases in the receiver's allocation induces more drastic action adjustments to capitalize on higher quality information, assured by the (already) high quality of information. Combined, these imply that the receiver displays increasing returns in their allocation.

The intuition above is most easily observed by revisiting the binary-state, quadratic loss utilities example provided in the introduction. Figures (1) and (2) illustrate how a receiver's payoff changes from an increase in her allocation. Starting from some $x_l \in (0,1)$, the receiver's payoff $U(x_l)$ can be found as a weighted average of her conditional expected payoff, $u(q)$ evaluated at beliefs $q_h(x_l)$ and $q_l(x_l)$.

Consider increasing the receiver's allocation to $x_l + \delta$, $\delta \in (0, 1-x_l)$. Fixing her actions at those taken prior to the increase in allocation, one first identifies their new conditional expected payoff by (i) obtaining the value of the tangent to $u(q)$ at either of the old beliefs, under either of the new beliefs $q_h(x_l+\delta)$ and $q_l(x_l+\delta)$, and (ii) taking a weighted average of the two. Labelling this payoff as $\tilde{U}(x_l, \delta)$, the difference $\tilde{U}(x_l,\delta) - U(x_l)$ is known as the accuracy benefit. Of course, the receiver also adjusts her actions such that her true payoff is $U(x_l+\delta)$, given by taking a weighted average of $u(q)$ evaluated at either of the new beliefs. The gain $U(x_l+\delta) - \tilde{U}(x_l,\delta)$, is known as the adjustment benefit. Finally, one observes that

$$\tilde{U}(x_l,\delta) - U(x_l) = \frac{\delta x_l}{16}, \quad U(x_l+\delta) - \tilde{U}(x_l,\delta) = \frac{\delta(\delta + x_l)}{16}$$

Both of which, fixing the marginal increase δ, are increasing in the starting allocation x_l. That is, consequent increases in the receiver's allocation increase the magnitude of both the accuracy and adjustment benefits.

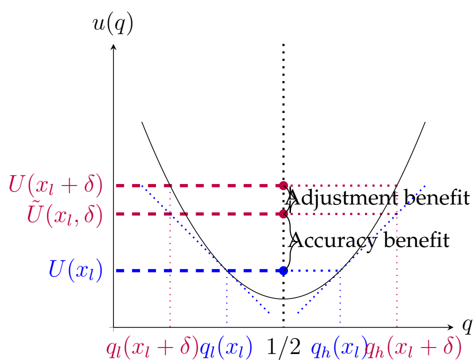

Figure 1: Payoff as a function of beliefs

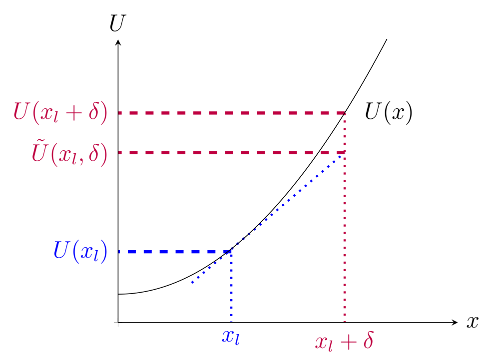

Figure 2: Payoff as a function of allocation

Naturally, the findings above are predicated on several core assumptions, of which we relax (and explore) in Section 4. Before proceeding on to such extensions, we make three additional remarks regarding the findings of Proposition 1.

First, observe that the discussion above relies very minimally on the current choice of primitives, and does not rely on the assumption of symmetry.

Second, due to receivers' increasing returns from being targeted, Sender always possesses an innate incentive to choose highly targeted allocations. As we will show in later sections, this is true even in the presence of various sources of diminishing returns to targeting, e.g. from production technologies (Section 4.1), persuasion concerns (Section 4.2.2) and strategic interactions (Section 4.3). In our current framework, this immediately implies that a utilitarian Sender fully targets a single receiver.

Corollary 1. *A fully targeted allocation is optimal for Sender.*

Finally, through inducing the receiver to tailor her actions towards each belief, an increase in the allocation x_i increases the dispersion of actions by each receiver. This is most easily observed when $A = [0, 1]$ and the receiver's optimal action is linear in the posterior mean, which we discuss next

Corollary 2. *Suppose that $A = [0, 1]$, and a receiver's optimal action is linear in the posterior mean, i.e., $a^*_m(x_i) = C_0 \int_{\text{supp}(F)} \theta[\frac{(x_i[g(m|\theta)-g(m)]+g(m))}{g(m)}]dF(\theta) + C_1$, where $C_0, C_1 \in \mathbb{R}$. Let $H(a|x_i)$ denote the distribution over actions for a (representative) receiver obtaining allocation x_i, i.e.,*

$$H(a|x_i) = \int_{\{m:a^*_m(x_i)\leq a\}} g(m)dm \tag{4}$$

Then, for all $0 \leq x'_i < x_i \leq 1$, $H(a|x_i)$ is a mean-preserving spread of $H(a|x'_i)$.

Corollary 2 illustrates another key difference between allocations of information versus physical goods. Whereas deterministic allocations of goods induce ex-post deterministic choices/actions by receivers, deterministic allocations involving information instead induce ex-post random choices of actions. Additionally, increasing a receiver's allocation increases the dispersion of her distribution over actions. Hence, any *distributional* concerns (over actions) by both Sender and Receivers can now affect on Sender's choice of allocations. For instance, an inherent preference for less dispersed actions by Sender, or when receivers prefer for other receivers' actions to be less-dispersed, may induce Sender into choosing a non-fully targeted allocation. We return to these points in sections 4.2 and 4.3.

4 Extensions

The basic framework suggests a strong (theoretical) incentive for the Sender to target her allocation at one receiver. This is driven by the convexity of receivers' payoffs in their allocation. In this section, we test the robustness of this conclusion by pursuing extensions which relax several core assumptions.

While we aim to provide a general discussion throughout, we will make the following assumption throughout this Section to simplify exposition.

Assumption 1. *$U(x_i)$ is strictly convex, twice-differentiable and satisfies $U(0) = 0$.*

Furthermore, it will often be useful to return to a tractable example for concrete results. One such example is the quadratic loss framework, normalized such that $U(0) = 0$, which also satisfies Assumption 1.

Definition 2. *We say that the receiver's payoff is of the **quadratic loss form** if $u(\theta, a) = -(\theta - a)^2 - (\max_{a \in A} \int_F [-(\theta - a)^2] dF(\theta))$ and $A = [0, 1]$*

Unlike Assumption 1, we will explicitly mention whenever our findings apply specifically to the case of the quadratic loss form.

4.1 General production functions

We have assumed under (1) that the Sender's choice set involves convex mixtures of signal mappings which are linear in Sender's allocation. Such an assumption is most relevant under the interpretation of Sender's allocation as the proportion of a report dedicated towards conveying information about receiver i's state, or, under the information production interpretation, when production technologies display "constant" returns to informativeness.

We now allow for more general mappings between Sender's allocation and receivers' composite signal. More precisely, suppose that

$$G^{x_i} = \varepsilon(x_i)G_i + (1 - \varepsilon(x_i))\underline{G}_i \tag{5}$$

Where we refer to $\varepsilon(x_i)$ as the *production function* of information. We impose two assumptions on $\varepsilon(x_i)$. First, we maintain that an increase in x_i increases the informativeness of Sender's signal to receiver i. This requires $\varepsilon(x_i)$ to be an increasing, continuously differentiable function of x_i. Second, we impose boundary conditions of $0 \leq \varepsilon(0) < \varepsilon(1) \leq 1$. Notice that we may have $\varepsilon(0) > 0$, i.e., a receiver may be partially informed even without intervention by Sender.

There are several scenarios where (5) applies. Interpreting Sender's allocation as information production, having $\varepsilon(x_i)$ be concave, convex or linear in x_i accommodates for production technologies with decreasing, increasing or constant returns to informativeness. Non-linear production can also result from additional noise induced through changes in Sender's allocation. Finally, $\varepsilon(0) > 0$ allows for relaxations of the scarcity constraint.

We now discuss general consequences of (5). First, when $\varepsilon(x_i)$ is convex in x_i such that there are increasing returns to informativeness, then $U(x_i)$ becomes even more convex in x_i. This simply enhances a receiver's increasing returns in their own allocation. Thus, $V(x) = \sum_{i=1}^{N} U(x_i)$ is convex in x, implying that fully target a single receiver is optimal.

Proposition 2. *Suppose that each receiver's signal is given by (5). If $\varepsilon(x_i)$ is convex in x_i, then a fully targeted allocation is optimal for Sender.*

Meanwhile, if $\varepsilon(x_i)$ is (strictly) concave in x_i, then two conflicting effects play a role in determining Sender's optimal allocation. To see this, let $U_L(x_i)$ denote the receiver's payoff when $\varepsilon(x_i) = x_i$. Then, $U(x_i) = U_L(\varepsilon(x_i))$. Twice-differentiating yields

$$U''(x_i) = \underbrace{\varepsilon''(x_i)U_L'(\varepsilon(x_i))}_{\text{Decreasing returns in production; } \leq 0} + \underbrace{[\varepsilon'(x_i)]^2 U_L''(\varepsilon(x_i))}_{\text{Increasing returns in consumption; } >0}$$

From a *production* standpoint, the marginal gains to signal informativeness are largest when the receiver's allocation is low, and smallest when the receiver's allocation is high. This naturally induces Sender towards choosing a more evenly spread allocation. From a *consumption* standpoint, receivers still exhibit increasing returns in the use of information. That is, $U(x_i)$ is convex in the net allocation $\varepsilon(x_i)$. As discussed in Section 3, this induces Sender towards a targeted allocation. The combination of such effects may therefore leave $V(x)$ neither concave nor convex, which occurs when neither effect is dominant. When this holds, Sender can find it optimal to choose an asymmetric, yet non-fully targeted allocation.

In the latter case, a general closed form characterization of the set of optimal allocations is not possible. We can, however, consider how changes in the degree of Sender's targeting change with respect to changes in features of the production function. Two examples are considered below.

First, suppose that $\varepsilon(x_i) = x_i^{1-\rho}$. Here, an increase in $\rho \in (0,1)$ increases the degree of diminishing returns to production. As per the discussion above, this should intuitively reduce the degree of targeting by Sender.

Second, suppose that $\varepsilon(x_i) = \rho + (1-\rho)\tilde{\varepsilon}(x_i)$, where $\tilde{\varepsilon}(x_i)$ is strictly increasing, concave, twice-differentiable and satisfies $0 \leq \tilde{\varepsilon}(0) < \tilde{\varepsilon}(1) \leq 1$. Here, an increase in ρ weakens the scarcity constraint, increasing each receiver's minimum endowment of information while leaving less room for discretion in the allocation to Sender. The former effect raises the marginal gain from further allocating information to *any* receiver, while the latter effect reduces the marginal gain (from Sender's end) from targeting *one* receiver (particularly given the concavity of $\tilde{\varepsilon}(x_i)$). The combination of both effects should encourage Sender to choose a more spread out allocation. That is, relaxing the scarcity constraint not only exogenously reduces the inequality of payoffs between receivers, but also *endogenously* by inducing a less targeted allocation by Sender.

Our next result confirms both prior intuitions, provided that the receiver's increasing returns to consumption is not too strong. To state it, recall $U_L(x_i)$ denotes the receiver's payoff under the benchmark case in Section 2, when $\varepsilon(x_i) = x_i$.

Proposition 3. *Suppose U_L is thrice-differentiable, $U_L'''(x_i) \leq 0$ such that increasing returns to consumption is not too strong, and for $\rho \in [0, 1)$, $\varepsilon(x_i)$ is equal to either of:*

1. ***Greater diminishing returns:*** $\varepsilon(x_i) = x_i^{1-\rho}$
2. ***Relaxing scarcity:*** $\varepsilon(x_i) = \rho + (1-\rho)\tilde{\varepsilon}(x_i)$, *$\tilde{\varepsilon}$ is twice-differentiable, strictly increasing and concave in x_i and $0 \leq \varepsilon(0) < \varepsilon(1) \leq 1$*

Denote Sender's payoff given ρ by $V(x, \rho)$. Then, $\overline{X}^*(\rho) = \arg\max_{x \in X} V(x, \rho)$ is non-increasing with respect to $\succ^M$ in ρ. That is, an increase in ρ cannot induce Sender into having her allocation become more targeted.

We conclude by discussing the tractable case where the receiver faces the quadratic loss form, i.e., given by Definition 2. If so, a receiver's payoff is

$$U(x_i) = (\varepsilon(x_i))^2 \int_0^1 \bigg(\mathbb{E}[\theta|m, 1] - \mathbb{E}[\theta]\bigg)^2 g(m)dm \tag{6}$$

Where $\mathbb{E}[\theta|m, 1] = \int_{\text{supp}(F)} \theta \frac{g(m|\theta)}{g(m)} dF(\theta)$ and $\mathbb{E}[\theta] = \int_{\text{supp}(F)} \theta dF(\theta)$. This yields simple sufficient conditions for fully targeting and evenly-spread allocations to be optimal for Sender.

Corollary 3. *Suppose that the receiver faces the quadratic loss form, and each receiver's signal is given by (5). Then,*

1. *If $(\varepsilon(x_i))^2$ is convex, a fully targeted allocation is optimal for Sender.*
2. *If $(\varepsilon(x_i))^2$ is concave, an evenly spread allocation is optimal for Sender.*

4.2 Different Sender objectives

So far, we have assumed that Sender aims to maximize the efficiency of information use, i.e., by maximizing the sum of receivers' payoffs. As discussed in Section 2.3, this primarily accommodates the interpretation of Sender as a purely utilitarian social planner.

It is also natural to expect Sender to have preferences which extend beyond efficiency. For instance, a government may be also be concerned about the inequality in the welfare of citizens when deciding how to allocate information. The government can also have ulterior "persuasion" motives over what actions citizens take, apart from the benefit that citizens obtain, from being allocated information.

We now aim to understand how non-efficiency related preferences may affect Sender's allocation of information. To do so, we investigate two relaxations of the assumption of a purely utilitarian Sender. First, we maintain the interpretation of Sender as a social planner, but now allow for Sender to possess preferences over ex-ante *equality* between receivers. Second, we consider the case where Sender has explicitly preferences over the actions taken by receiver. That is, when Sender's allocation information with a partial *persuasion* motive.

4.2.1 Egalitarian Sender

We begin with egalitarian Sender who cares both about the efficiency of allocations, and the equality of receivers' payoffs. Sender's optimal allocation now is influenced by two competing forces. On one hand, equality concerns originating from Sender's preferences result in diminishing returns towards targeted allocations. This induce Sender into choosing a more evenly spread allocations. Meanwhile, efficiency concerns originating from Receivers' use of information result in increasing returns towards targeted allocations. This induce Sender into choosing a more targeted allocation.

We illustrate these trade-offs with the following example. Suppose that Sender's objective function is now

$$V(x) = \frac{\beta}{N}\sum_{i=1}^{N} U(x_i) + (1-\beta)\min_i U(x_i), \quad \beta \in [0,1] \tag{7}$$

Observe that a greater value of β implies that Sender places greater emphasis on maximizing efficiency, $\sum_{i=1}^{N} U(x_i)$, over minimizing inequality, $\min_i U(x_i)$. With respect to (7), we find the following

Proposition 4. *Suppose $V(x)$ is as defined in (7). Define $\overline{\beta}(N) = \frac{U(1)-U(0)}{N(U(\frac{1}{N})-U(0))}$. Then, for all $\beta > \overline{\beta}(N)$, the fully targeted allocation is strictly optimal for Sender. Meanwhile, for all $\beta < \overline{\beta}(N)$, the evenly spread allocation is strictly optimal for Sender.*

Proposition 4 yields two key insights. First, Sender may continue choosing a highly asymmetric allocation even while having a (positive) preference for equality. Indeed, $\overline{\beta}(N) > 0$ always holds as $U(1) > U(0)$. Thus, Sender transitions to the equal allocation only when her preferences for equality are sufficiently high. The more important insight is that it is almost always the case that either one of maximally-spread (fully-targeting) or minimally-spread (fully-mixed) allocations are optimal. This stems from receivers' increasing returns to the use of information, which implies that all other allocations are either insufficiently efficient or insufficiently equal. As a result, the transition from a preference for efficiency to a preference for equality can lead to sudden transitions in the way information is allocated.[9]

One can also consider how changes in the heterogeneity in the receiver population's needs affects the way Sender allocates information. As one may always rescale the mass of receivers, this is simply captured by an increase in N. Intuitively, with a greater diversity in needs, Sender stands to lose more by targeting her allocation towards a select few needs, e.g. by choosing a fully targeted allo-

[9]Sudden transitions can also hold for other welfare specifications. For instance, consider the case of quadratic loss and the separable isoelastic welfare function $V(x,\beta) = \sum_{i=1}^{N}[U(x_i)]^{\beta}$, $\beta \in [0,1]$. Using (6), $V(x) = K\sum_{i=1} x_i^{2\beta}$, where $K > 0$. Thus, any change from $\beta > 1/2$ to $\beta < 1/2$ causes Sender's optimal allocation to transition from fully targeting to the equal allocation.

cation. Thus, one may expect an increase in N to push Sender towards a more even allocation. We confirm that the intuition holds when Sender's payoffs are provided by (7).

Corollary 4. *Suppose $V(x)$ is as defined in (7). Then, $\overline{\beta}(N) = \frac{U(1)-U(0)}{N(U(\frac{1}{N})-U(0))}$ is strictly increasing in N.*

4.2.2 Persuasion-focused Sender

We now move on to the case where Sender's has explicit preferences over receivers' actions. More precisely, suppose that Sender's objective function is now given by

$$V(x, \beta) = \beta \sum_{i=1}^{N} U(x_i) + (1-\beta) \sum_{i=1}^{N} \int_0^1 w(a_m^*(x_i))g(m)dm \tag{8}$$

Where A is a compact real interval, $w : A \to \mathbb{R}$ is twice-differentiable and $\beta \in [0, 1]$. Observe that whenever $\beta > 0$ holds, Sender cares both about receivers' payoffs, and the actions that they take. Additionally, an increase in β increases Sender's emphasis on allocating for the sake of *persuasion*, i.e., maximizing the second term of (8), rather than for the benefit of receivers, i.e., maximizing the first term of (8).

Preferences of the form described in (8) arise for two main reasons. Sender herself may care not only about the benefit that receivers obtain from information, but also the way in which receivers act. Receivers themselves may also be affected by externalities generated by the actions of other receivers. When a receiver's payoffs are of the quasilinear form $\beta u(\theta_i, a_i) + (1-\beta)\sum_{j \neq i} w(a_j)$, i.e., externalities have no strategic impact on the receiver's choice of actions, a utilitarian Sender's payoff may be written in the form of (8).

Given that Sender only controls the quality of information, i.e., level of informativeness, rather than explicitly designing the type of information available to receivers, a characterization of optimal allocations for general $w(\cdot)$ is out of reach. There is, however, one important case that our analysis does account for, which we discuss next.

Recall from Corollary 2 that when a receiver's action is linear in the posterior mean, Sender's allocation directly affects the spread of the distribution over actions for each receiver. In particular the change in the spread is "linear", given the linearity of the posterior mean in x_i. This suggests that convexity, i.e., a preference for maximizing spread, or concavity, i.e., a preference for minimizing spread, of $w(\cdot)$ are sufficient for monotone changes in the degree of Sender's targeting from a change in β. Our next result, which uses the quadratic loss case such the receiver's optimal action is the posterior mean, confirms this intuition.[10]

Proposition 5. *Suppose the receiver faces the quadratic loss form, and that Sender's payoff is given by (8). Let $X^*(\beta) = \arg\max_{x \in X} V(x, \beta)$.*

1. *If $w(a)$ is convex in a, then for all $\beta \in [0, 1]$, fully targeting is optimal for Sender.*

2. *If $w(a)$ is concave in a, then $X^*(\beta)$ is non-decreasing with respect to $\succ_w^M$ in β. That is, an increase in persuasion preferences β cannot induce Sender into allocating more information and having her allocation be more targeted.*

Proposition 5 connects changes in Sender's allocation from changes in her focus on persuasion to preferences over the distribution over actions. Here, a convex $w(\cdot)$ captures increasing returns towards inducing a greater dispersion over actions. As a result, a persuasion-focused Sender optimally fully targets a single receiver. If so, then Sender's preferences over efficiency and persuasion agree, and thus Sender fully targets a single receiver for any β. Conversely, a concave $w(\cdot)$ captures strong preferences towards inducing a smaller dispersion over actions. As a result, Sender's preferences over efficiency and persuasion disagree, and so a Sender which is sufficiently focused on persuasion opts to minimize the spread over actions for a single receiver. In particular, when β is sufficiently small and $w(\cdot)$ strictly concave, then Sender chooses the allocation which induces the least dispersed distribution over actions among all receivers. This coincides with choosing no disclosure.

[10]While we maintain the use of quadratic loss here, one can verify that the proposition holds when the receiver's optimal action is a linear combination of posterior moments.

We conclude by comparing the findings of Proposition 5 to insights from (i) traditional physical goods allocation problems, and (ii) the information design literature. First, with physical goods where the ex-post effect of allocations is often deterministic, the sign of $w(\cdot)$ plays a substantial role. In our problem, the sign of $w(\cdot)$ does not play a substantial role in determining how Sender allocates information - only its curvature.

Meanwhile, it is well known that when a Sender's payoff is convex (concave) in beliefs, she has an incentive to choose maximally-spread (minimally-spread) distributions over posteriors (e.g. Kamenica and Gentzkow, 2011, Remark 1). As a result, full disclosure (no-disclosure) is optimal for Sender. In our environment, since the posterior mean is linear in Sender's allocation, a convex $w(a)$ (concave $w(a)$) implies that her persuasion payoffs are convex (concave) in her allocation. Thus, a greater motive for persuasion (weakly) induces Sender towards a maximally-spread (minimally-spread) allocations. This is achieved by fully targeting a single receiver (no disclosure).

4.3 Strategic interactions

The basic model assumes that one receiver's actions has no effect on the choice of actions of another receiver. Alongside the assumption of independent states, this primarily accommodates the interpretation of receivers being spatially segregated, such that one receiver's actions (and states) has no impact on the other.

Such assumptions do not apply when strategic interactions play a role in the receiver's decision-making. Given our existing focus on an information allocation problem, we are most interested when such strategic interactions directly conflict with a receiver's use of information. That is, when the "smart" action, i.e., the action which efficiently utilizes information obtained, conflicts with the "strategically optimal action", i.e., given how other receivers are acting. These effects change the marginal gains from allocating receivers information.

Motivated by such concerns, we first introduce a general framework to incorporate strategic interactions. We shall assume the receiver faces the quadratic loss form.

Extended model Let a receiver's payoff given action profile $\boldsymbol{a} = (a_i, a_{-i})$, where $a_{-i} = (a_j)_{j \neq i}$ denotes the actions of other receivers, be equal to

$$u(\theta_i, \boldsymbol{a}, \phi) = -(\theta_i - a_i)^2 + \phi s(\boldsymbol{a})$$

Observe that $-(\theta_i - a_i)^2$ captures the component of the receiver's payoff with varies with information, but is independent of strategic effects. Meanwhile, $s(\boldsymbol{a})$ captures the component of the receiver's payoff with varies with other receivers' actions, but is independent of the informativeness of Sender's signal. A higher $\phi \in [0, \overline{\phi}]$, $\overline{\phi} > 0$ captures an environment where the strategic interactions become more important to a receiver. We further assume that $s(a)$ is strictly concave in a_i, and twice-differentiable in $\boldsymbol{a}$. To simplify exposition, we suppose that for all $\theta_i \in \Theta$, $\boldsymbol{a} \in [0,1]^N$ and $\phi \in [0, \overline{\phi}]$,

$$\phi \frac{\partial^2 s(\boldsymbol{a})}{\partial a_i^2} + \phi \sum_{j \neq i} \left| \frac{\partial^2 s(\boldsymbol{a})}{\partial a_i \partial a_j} \right| < 2 \tag{9}$$

Under these assumptions, there exists a unique (Bayesian-Nash) equilibrium in actions between receivers for any $x \in X$ and $\boldsymbol{m} \in M^N$ observed by the receivers, in stage 3.[11] Denote the equilibrium actions as $\boldsymbol{a}^*_{\boldsymbol{m}}(x) = (a^*_{1,\boldsymbol{m}}(x), ..., a^*_{N,\boldsymbol{m}}(x))$, which we assume lies in the interior of $[0,1]^N$.

Let $U^i(x)$ denote receiver i's payoff, which is now dependent on the full vector of allocations and equal to

$$U^i(x,\phi) = \int_{\boldsymbol{m} \in M^N} \left[\int_{\text{supp}(F)} \begin{pmatrix} -(\theta - a^*_{i,\boldsymbol{m}}(x))^2 \\ +\phi s(\boldsymbol{a}^*_{\boldsymbol{m}}(x)) \end{pmatrix} \left(\frac{x_i[g(\theta|m_i) - g(m_i)] + g(m_i)}{g(m_i)} \right) dF(\theta) \right] \prod_j g(m_j) d\boldsymbol{m} \tag{10}$$

Where the integral is taken with respect to the vector $\boldsymbol{m} = (m_1, ..., m_N)$. Extend-

[11]Existence is guaranteed by each receiver's payoff being strictly concave in their own action. Uniqueness is guaranteed by the contraction condition (9), which ensures that the best-reply map for all receivers is a contraction under any allocation and message vector (e.g. see Vives (1999)).

ing our solution concept, we now look for optimal allocations which maximize $V(x,\phi) = \sum_{i=1}^{N} U^i(x,\phi)$: Sender's payoff given receivers play the unique equilibrium under any vector of messages and allocation.

With strategic interactions, changes in Sender's allocation direct affect receivers' ability to take more informed actions on each state, and indirect affect the behaviour of other receivers through the change in equilibrium. More precisely,

$$\frac{\partial V(x,\phi)}{\partial x_i} = \underbrace{\int_{\boldsymbol{m}\in M^N} \left(\int_{\text{supp}(F)} -(\theta - a^*_{i,\boldsymbol{m}}(x))^2 \left(\frac{g(\theta|m_i) - g(m_i)}{g(m_i)}\right) dF(\theta) \right) \prod_i g(m_i) d\boldsymbol{m}}_{\text{(i): informational benefit}}$$
$$+ \underbrace{\phi \sum_{j=1}^{N} \int_{\boldsymbol{m}\in M^N} \left(\sum_{k\neq j} \frac{\partial a^*_{k,m}(x)}{\partial x_i} \frac{\partial s(a^*_m(x))}{\partial a_k} \right) \prod_i g(m_i) d\boldsymbol{m}}_{\text{(ii): strategic effect}}$$

Observe that the marginal gain from allocating information to receiver i can be split into two components. (i) captures the *informational benefit* of receiver i from being allocated better information about their state. This arises from receiver adjusting their actions to better utilize information obtained (to their benefit). (ii) captures the indirect *strategic effect* on the change in the equilibrium actions of all (other) receivers on a receiver's payoff. This affects the payoffs of all receivers. Akin to Section 4.2.2, given the stochastic nature of messages observed (and hence equilibrium actions taken), the net benefit/loss to (other) receivers depends heavily on the curvature of $s(\cdot)$.

An increase in ϕ, the salience of strategic interactions, shifts a receiver's focus towards choosing actions which are "strategically optimal". To understand how this can affect Sender's allocation, fix Sender choosing a highly targeted allocation. On one hand, the targeted receiver can be increasingly reluctant to adjust their action by too much in response to information obtained. This is as it can induce an unfavourable action choice from other receivers, which affects him greatly through the now larger term $\phi s(\boldsymbol{a}^*_{\boldsymbol{m}}(x))$. This weakens the channel through which increasing returns from targeting the receiver materializes.

Meanwhile, consider other non-targeted receivers. As the magnitude of the strategic effect (ii) begins to dominate, these receivers now more drastically adjust their actions in response to other receivers' actions. Under a targeted allocation, which induces (relatively) more drastic adjustments of actions by targeted receivers, these receivers can now be induced into taking actions highly unfavourable given their true state. This also increases the (ex-ante) spread over actions taken by non-targeted receivers in equilibrium, which can be unfavourable to all parties at an ex-ante level when $s(\cdot)$ is concave enough.

Combined, the discussion above suggests two conditions on s under which an increase in ϕ induces less targeted or even no allocation by Sender. First, when s captures strategic concerns which directly conflict with receivers' use of information. Second, when at the ex-ante level, s captures receivers' sufficient dislike for dispersed allocations in equilibrium. We illustrate such concepts in two simple environments, focusing on a two-receiver case for tractability.

Conformity Suppose that there are two receivers, $i = 1, 2$, and that each have payoffs given by

$$u_i(\theta_i, \boldsymbol{a}, \phi) = -(\theta_i - a_i)^2 - 2\phi s(a_j - a_i) \tag{11}$$

Where $j \neq i$, s is fourth-differentiable ($s \in \mathcal{C}^4[-1, 1]$), strictly convex and symmetric about 0. Here, the strategic term captures the loss experienced by a receiver from deviating from other receivers' behaviour (second term).

By the discussion above, one suspects that for a sufficiently convex s, i.e., such that the strategic term is sufficiently concave in actions, that an increase in ϕ weakly induces less targeted allocations and a lower total allocation by Sender. Our next result confirms this intuition.

Proposition 6. *Suppose that receivers' payoffs are given by (11). Define* $f_\phi(\Delta) = s(\Delta)(1 + 2\phi w''(\Delta)) - \phi(s'(\Delta))^2$ *and* $g_\phi(\Delta) = 1 + 2\phi s''(\Delta)$*. Suppose that for all*

$\Delta \in (-1,1)$ *and* $\phi \in [0,\overline{\phi}]$, $s''' \geq 0$, $\frac{f_\phi(\Delta)}{g_\phi(\Delta)}$ *is convex in* Δ, *and*

$$\frac{f''_\phi(\Delta)[g_\phi(\Delta)]^2 - 3f'_\phi(\Delta)g_\phi(\Delta)g'_\phi(\Delta) - f_\phi(\Delta)g_\phi(\Delta)g''_\phi(\Delta) + 3f_\phi(\Delta)(g'_\phi(\Delta))^2}{[g_\phi(\Delta)]^3} \geq 0 \tag{12}$$

holds, where the derivatives are taken with respect to Δ. *Then,* $X^*(\phi) = \arg\max_{x\in X} V(x,\phi)$ *is non-increasing with respect to* $\succ_w^M$ *in* ϕ.

One-sided biases Next, suppose that there are two receivers, $i = 1, 2$, and that each have payoffs given by

$$u_i(\theta_i, \boldsymbol{a}, \phi) = -(\theta_i - a_i)^2 + 2\phi s(a_i + a_j) \tag{13}$$

Where $j \neq i$, s is fourth-differentiable ($s \in \mathcal{C}^4[-2,2]$), symmetric about 0 and strictly concave. Here, receivers are induced into reducing their actions. In this sense, s induces one-sided biases which can directly conflict with the informative content of messages. Again, one naturally expects a sufficiently concave s to result in an increase in ϕ to induce less targeted and a lower total allocation by Sender. This is when one receiver catering to their biases sufficiently lowers the marginal benefit from another from doing the same. We justify this intuition with the following result.

Proposition 7. *Suppose that receivers' payoffs are given by (13). Define* $\tilde{f}_\phi(\nabla) = s(\Delta)(1 - 2\phi w''(\nabla)) + (\phi/2)(s'(\nabla))^2$ *and* $\tilde{g}_\phi(\nabla) = 1 - 2\phi s''(\nabla)$. *Suppose that for all* $\nabla \in (0,2)$ *and* $\phi \in [0,\overline{\phi}]$, $s''' \leq 0$, $\frac{\tilde{f}_\phi(\Delta)}{\tilde{g}_\phi(\Delta)}$ *is concave in* Δ, *and*

$$\frac{\tilde{f}''_\phi(\nabla)[\tilde{g}_\phi(\nabla)]^2 - 3\tilde{f}'_\phi(\nabla)\tilde{g}_\phi(\nabla)\tilde{g}'_\phi(\nabla) - \tilde{f}_\phi(\nabla)\tilde{g}_\phi(\nabla)\tilde{g}''_\phi(\nabla) + 3\tilde{f}_\phi(\nabla)(\tilde{g}'_\phi(\nabla))^2}{[\tilde{g}_\phi(\nabla)]^3} \leq 0 \tag{14}$$

holds, where the derivatives are taken with respect to ∇. *Then,* $X^*(\phi) = \arg\max_{x\in X} V(x,\phi)$ *is non-increasing with respect to* $\succ_w^M$ *in* ϕ.

We conclude by emphasizing strong similarities between Proposition 5, i.e., involving persuasion preferences, and the results of Propositions 6 and 7, i.e., involving strategic interactions. Both cases highlight the importance of considering the curvature of the non-information term (with respect to actions) in an information allocation problem. When the term is sufficiently concave in actions, such that minimizing spread over actions is important, Sender is induced into choosing a less targeted and possibly lower total allocation.[12]

4.4 Incomplete information

An important aspect of the increasing returns result of Proposition 1 is a receiver's ability to accurately adjust her actions in response to the quality of information allocated by Sender. This is as the actions the receiver takes are often sufficiently riskier than that taken at the prior (as they are catered towards a particular posterior belief), and are only taken under the assurance of higher quality information by the Sender.

When there is uncertainty about the quality of Sender's information source, there can be a disconnect between the receiver's belief about the quality of information provided, and that actually provided Sender. Notably, a receiver could be too optimistic, "overadjusting" their actions relative to the actual quality of information provided by Sender. As we will show in this section, this possibility has two key consequences. First, Senders possessing low-quality information may prefer equilibria involving a more balanced allocation of information among receivers, to avoid the cost of receivers overadjusting actions. Second, sufficiently low quality Senders may choose to opt out of disclosing any information.

Extended model We now suppose that Sender has a *type* t drawn from the unit interval $[0, 1]$. Let $Q \in \Delta[0, 1]$ be the common prior over Sender's types, which is absolutely continuous, has full support and satisfies $\int_0^1 t dQ(t) > 0$. Receivers do

[12]The key difference between the two lies in the effect of the term ϕ/β on a receiver's own use of information. Unlike the case with persuasion preferences, with strategic interactions, increasing ϕ directly reduces the receiver's *own* increasing returns in her allocation.

not observe Sender's allocation nor her type.[13] Given type t, the composite signal is now given by

$$G_t^{x_i} = x_i t G_i + (1 - x_i t)\underline{G}_i \tag{15}$$

Observe that a larger t implies that the underlying signal available to Sender is more informative to receivers but retains the same message space. Thus, receivers' source of uncertainty purely revolves around the *quality* of information observed.

In this environment, Sender may want to "explicitly" declare that she is not disclosing any information. We incorporate this as follows. Let $\sigma(t)$ denote the strategy of a Sender of type t, and let $\sigma = (\sigma(t))_{t\in[0,1]}$ be a strategy profile of Sender. Each Sender chooses from the set $\sigma(t) \in \{\emptyset\} \cup \Delta X$. A Sender who chooses $\emptyset$ "opts out" of sending information, leaving all receivers to observe a completely uninformative message $\chi \notin M$ with probability one. Meanwhile, choosing a $\sigma(t) \in \Delta X$ is akin to choosing a randomization over allocations in X, where the timing of the continuation game mirrors that in Section 2. Importantly, receivers are aware whenever Sender chooses not to send any information, such that they obtain their default payoffs of zero whenever the Sender opts out. Let $\mathcal{T}(\sigma) \subseteq [0, 1]$ denote the set of Sender types under σ who opt in. For all such senders, let $\sigma(x|t)$ denote the probability that allocation x is chosen.

One finds that the expected *effective allocation*, i.e., x_i scaled by Sender's types, which determines receivers' optimal action on each message under any sensible solution concept. This is as Senders are unable to signal their quality, given that for any allocation choice (apart from opting out), a receiver observes all messages in M with positive probability. Fixing σ, let $\overline{x}_i(\sigma)$ denote this expected effective allocation:

$$\overline{x}_i(\sigma) = \int_{t\in\mathcal{T}(\sigma)} \int_{x_i\in\text{supp}(\sigma_i(t))} \left(\frac{t x_i}{\int_{t\in\mathcal{T}(\sigma)} dQ(t)} \right) d\sigma_i(x_i|t) dQ(t)$$

[13] In the benchmark model of Section 2, given that Sender's signal quality and preferences are common knowledge, it does not matter whether receivers observe Sender's allocation. More precisely, if Sender's allocation is not observed, then under a (suitably defined) Sender-preferred equilibrium, Sender's payoff is identical to that identified in Section 2 for any allocation chosen.

Where we let $\overline{x}_i(\sigma) = 0$ for all $i = 1, ..., N$ in the extreme case where $\mathcal{T}(\sigma) = \emptyset$. From here, let $\hat{U}(x_i|\sigma, t)$ denote the payoff of a receiver under true allocation x by a type-t Sender, given expected effective allocation (scaled by Senders' types) $\overline{x}_i(\sigma)$. This is simply

$$\begin{aligned}\hat{U}(x_i|\sigma, t) &= \int_{m\in M}\left(\int_{\text{supp}(F)} u(\theta, a^*_m(\overline{x}_i(\sigma)))\Big(tx_i(g(m|\theta) - g(m)) + g(m)\Big)dF(\theta)\right)dm \\ &= x_i t\overline{U}(\overline{x}_i(\sigma)) + \underline{U}(\overline{x}_i(\sigma))\end{aligned}$$

Where $\overline{U}(x_i)$ captures the marginal gain from increasing the receiver's allocation, given that he adjusts their actions expecting allocation x_i, and $\underline{U}(x_i)$ capture the additional risk borne by a receiver from adjusting their actions expecting allocation x_i,[14] and are given by

$$\overline{U}(x_i) = \int_0^1 \int_{\text{supp}(F)} u(\theta, a^*_m(x_i))[g(m|\theta) - g(m)]dF(\theta)dm \tag{16a}$$

$$\underline{U}(x_i) = \int_0^1 \int_{\text{supp}(F)} u(\theta, a^*_m(x_i))g(m)dF(\theta) \tag{16b}$$

While the receiver's actions are determined by the expectation induced by σ, the true probabilities under which messages are observed on some state are determined by the Sender's allocation x_i and their type t. Further let $\hat{V}(x|\sigma, t) = \sum_{i=1}^N \hat{U}(x_i|\sigma, t)$.

We now define an appropriate incomplete information solution concept

Definition 3. *A Perfect Bayesian Equilibrium (PBE) consists of a mapping σ^* under which*[15]

1. *For all $t \in \mathcal{T}(\sigma^*)$, $\max_{x'\in X} \hat{V}(x'|\sigma^*, t) \geq 0$, and for all $x \in \text{supp}(\sigma^*(t))$, $x \in$*

[14] In Appendix B (Remark 2), we show that $\overline{U}(x_i) \geq 0$ and is strictly increasing in x_i, while $\underline{U}(x_i) \leq 0$ and is strictly decreasing in x_i.

[15] Our reduced-form definition implicitly requires that receivers' beliefs must be consistent with Sender's strategy, and that Receivers are best-responding to beliefs formed which are consistent with σ: this is a receiver chooses actions to maximize their (conditional) expected payoff given $\overline{x}_i(\sigma)$.

$\arg\max_{x' \in X} \hat{V}(x'|\sigma^*, t)$.

2. *For all* $t \notin \mathcal{T}(\sigma^*)$, $0 \geq \max_{x' \in X} \hat{V}(x'|\sigma^*, t)$.

That is, each Sender who chooses to opt in allocates optimally fixing receivers' beliefs induced by σ^*, and weakly prefers allocating information over opting out. The opposite holds for Senders who choose not to opt in. We further observe that (abusing notation) under any PBE σ^*, a Sender's payoff under PBE σ^* is given by

$$\hat{V}(\sigma^*|t) = \begin{cases} \int_{x \in \text{supp}(\sigma^*)} \hat{V}(x|\sigma, t) d\sigma^*(x), & \sigma^* \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

One finds that there can exist many uninformative PBE under which no information is allocated to receivers.[16] Our goal is to discuss PBE where some types of Sender prefer to allocate information amongst receivers. More precisely, we draw focus to the following types of PBE:

Definition 4. *A PBE* σ^* *is informative if*

1. $\overline{x}_i(\sigma^*) > 0$ *for at least one* i.

2. *There exists a* $t \in [0, 1]$ *under which* $\int_{\text{supp}(\sigma^*)} \hat{V}(x|\sigma^*, t) d\sigma^*(x|t) > 0$. *That is, at least one type of Sender strictly benefits from allocating information (and thus proceeds to do so) over opting out.*

3. $\sigma^*(t) = \emptyset$ *if and only if* $\max_{x' \in X} \hat{V}(x'|\sigma^*, t) < 0$.[17]

The crucial insight is that fixing the expected effective allocation under σ^* and receivers' best-response to such an expectation, a type-t Sender's payoff is now *linear* in her actual allocation of x_i. This is as a change in any Sender's allocation, fixing the receivers' beliefs, has no impact on their choice of action. In

[16] A uninformative PBE under which $\sigma^*(t) = \emptyset$ for all $t \in [0, 1]$ always exists. Here, $V(x|\sigma^*, t) = 0$ for all $x \in X$ and $t \in [0, 1]$ (given $t_i(\sigma^*) = 0$ for all $i = 1, ..., N$), and so opting out is a best-response for every type of Sender.

[17] That is, we break indifferences in favour of sending information to receivers.

this sense, uncertainty alters the allocation problem in two aspects, changing both the marginal gains to the precision of allocated information (as different types of Sender have different quality of information), and also entirely stopping a Sender from controlling the channel through which the benefits of information manifest - the adjustment in actions.

The observation above has two key implications. First, every PBE which has a positive measure of types of Sender allocating information to at least one Receiver, must also have a positive measure of (low-) types of Sender choosing to opt out. This is as receivers take actions which are "too-risky" relative to the actual quality of information offered by such Senders. The very aspect of an information allocation problem, i.e., the way receivers adjust their actions in anticipation of Sender's allocation, which led to increasing returns in the complete information case, now works *against* Sender's interests. Thus, fearing overreaction by receivers, such Senders opt out of disclosing information.

Lemma 1. *Take any informative PBE σ^*. Then, there exists a threshold type of Sender, $\bar{t}(\sigma^*) \in (0,1]$, such that $\mathcal{T}(\sigma^*) = [\bar{t}(\sigma^*), 1]$. That is, a Sender opts in if and only if the quality of information she provides is sufficiently high, and opts out otherwise.*

Second, since Sender's payoff is now linear in her actual allocation of x_i, every Sender who opts in must only be allocating information amongst the receivers with the best use of information, i.e., the receivers in $\arg\max_i \overline{U}(\overline{x}_i(\sigma^*)))$.

Lemma 2. *Take any informative PBE σ^*, and suppose there exists two receivers i, j under which $\overline{U}(\overline{x}_i(\sigma^*)) > \overline{U}(\overline{x}_j(\sigma^*))$. Then, $i \in \arg\max_k \overline{U}(\overline{x}_k(\sigma^*))$ and $\overline{x}_j(\sigma^*) = 0$. Hence, for all i, j under which $\overline{x}_i(\sigma^*) > 0$ and $\overline{x}_j(\sigma^*) > 0$, $\overline{x}_i(\sigma^*) = \overline{x}_j(\sigma^*)$.*

Lemma 2 implies that all PBE involve receivers being split into two groups: "targeted receivers" who receiver an equal split of the allocation by high-enough type Senders, and "untargeted receivers" who are never allocated any information. This rules out the possibility for equilibria under which receivers whom are allocated information obtain different expected effective allocations, but does not rule out the possibility for equilibria under which more than one receiver obtains a positive expected effective allocation. Indeed, a PBE under which *any* subset of receivers is allocated information always exists, which we show next

Proposition 8. *For any informative PBE σ^*, there exists a subset of receivers $\mathcal{N}(\sigma^*) \subseteq \mathcal{N} = \{1, ..., N\}$ such that*

1. *For all $i \in \mathcal{N}(\sigma^*)$, $\overline{x}_i(\sigma^*) = \frac{\int_{\overline{t}(\sigma^*)}^1 tdQ(t)}{(1-Q(\overline{t}(\sigma^*)))|\mathcal{N}(\sigma^*)|}$. Every such receiver obtains a payoff equal to*

$$(1 - Q(\overline{t}(\sigma^*)))U\left(\frac{\int_{\overline{t}(\sigma^*)}^1 tdQ(t)}{(1 - Q(\overline{t}(\sigma^*)))|\mathcal{N}(\sigma^*)|}\right)$$

2. *For all $i \notin \mathcal{N}(\sigma^*)$, receivers obtain a payoff of 0*

Additionally, for any $\mathcal{I} \subseteq \mathcal{N}$, there exists an informative PBE σ^* with $\mathcal{N}(\sigma^*) = \mathcal{I}$.

Proposition 8 allows us to establish two additional insights. First, we can now demonstrate that almost all types of Sender *strictly* prefer the complete information environment. To see this, take any PBE with $\mathcal{N}(\sigma^*)$ number of receivers allocated information with threshold type $\overline{t}(\sigma^*)$. Every receiver allocated information reacts (identically) in response to the effective allocation $\overline{x}_i(\sigma^*) = \frac{\int_{\overline{t}(\sigma^*)}^1 tdQ(t)}{(1-Q(\overline{t}(\sigma^*)))|\mathcal{N}(\sigma^*)|}$. Hence, a Sender's payoff from opting in is simply given by

$$\begin{aligned}
\hat{V}(\sigma^*|t) &= \sum_{i \in \mathcal{N}(\sigma^*(t))} \int_{x_i \in \text{supp}(\sigma_i^*(t))} \left(tx_i\overline{U}(\overline{x}(\sigma^*)) + \underline{U}(\overline{x}(\sigma^*))\right) d\sigma_i^*(x_i) \\
&= |\mathcal{N}(\sigma^*)|\left(\frac{t}{|\mathcal{N}(\sigma^*)|}\overline{U}(\overline{x}(\sigma^*)) + \underline{U}(\overline{x}(\sigma^*))\right)
\end{aligned}$$

Which is simply $|\mathcal{N}(\sigma^*)|$ times the value of the tangent to $U(x_i)$ at $x_i = \frac{\int_{\overline{t}(\sigma^*)}^1 tdQ(t)}{(1-Q(\overline{t}(\sigma^*)))|\mathcal{N}(\sigma^*)|}$ and evaluated at $\frac{t}{|\mathcal{N}(\sigma^*)|}$. Given the strict convexity of $U(x_i)$, the tangent lies below $U(x_i)$ for all $t \neq \overline{x}(\sigma^*)$. This suggests that almost all types of Sender are strictly better off with complete information, precisely due to the receiver's "incorrect" adjustment of actions relative to what is allocated by Sender.

Second, Proposition 8 finds the existence of many possible informative PBE. Among these, unlike the complete information environment, it is *not* the case that all Senders prefer the same full-targeting like PBE (i.e., when $\mathcal{N}(\sigma^*)$ is a singleton). In fact, we can show something stronger.

Corollary 5. *Take any informative PBE $\sigma^*, \widetilde{\sigma}^*$ under which $|\mathcal{N}(\sigma^*)| > |\mathcal{N}(\widetilde{\sigma}^*)|$. Then, there exists $0 \leq \hat{t}(\sigma^*) < \tilde{t}(\sigma^*) \leq 1$, both of which solve $V(\sigma^*|t) = |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{t}{|\mathcal{N}(\sigma^*)|})$, such that for all intermediate types $t \in (\hat{t}(\sigma^*), \tilde{t}(\sigma^*))$, $V(\sigma^*|t) > V(\widetilde{\sigma}^*|t)$.*

Corollary 5 exhibits the downsides of receivers "overreacting" to information when quality is uncertainty. There always exists a positive measure of *intermediate* types which prefer less targeted informative PBE, i.e., larger $|\mathcal{N}(\sigma^*)|$, over more-targeted PBE, i.e., smaller $|\mathcal{N}(\widetilde{\sigma}^*)|$. Intuitively, such intermediate-type Senders, due to having less precise information, prefer receivers to "overreact less". This is achieved in a PBE under which receivers obtain a smaller allocation: a larger $|\mathcal{N}(\sigma^*)|$.

5 Conclusion

The question at the heart of the *information design* literature is simple, but far-reaching: "how does a self-interested Sender persuade a Bayesian Receiver?" In this paper we ask a complementary question: "how does a benevolent Sender best provide information to multiple Receivers in the presence of communication constraints?" Our question speaks to range of policy-relevant issues, including those pertaining to public-health messaging.

In our baseline model we obtain a sharp characterization. Sender should target all available information to a single Receiver. This stems from the implicit convexity of Receiver's payoff in information. We have shown how modifications of the underlying environment can lead to more balanced information provision. These results have implications for understanding how close to optimal information provision is in various settings, and how it might be improved.

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Appendix

A1: Generalization

Here, we prove Proposition 1 in the context of a slightly more general environment through allowing for asymmetry.

Let Θ_i be the state space of receiver i, which we assume to be a compact metric space. The common prior over states in Θ_i is given by $F_i \in \Delta\Theta_i$, which we assume has full support. Each receiver has action space A_i, which we assume is a compact metric space. Let receiver i's payoff be denoted by $u_i(\theta_i, a_i)$, which we assume is continuous in (θ_i, a_i). Each receiver's signal is given by $(G_i^{x_i}, M_i)$, where M_i is a compact real interval, and

$$G^{x_i} = x_i G_i + (1 - x_i)\underline{G}_i$$

Where $G_i : \Theta_i \to \Delta M_i$ admits conditional density $g_i(m_i|\theta_i)$, and $\underline{G}_i$ is any signal mapping (with message space M_i) which admits a density $\underline{g}_i(m_i|\theta_i) = \underline{g}_i(m_i)$. The latter point implies all messages are drawn with equal frequency on all states, such that $\underline{G}_i$ is completely uninformative about receiver i's state.[18]

Letting $U_i(x_i)$ denote receiver i's payoff under allocation x (which still only depends on x_i), we now prove the slightly more general version of Proposition 1.

Proposition 9. *For each $i = 1, ..., N$, $U_i(x_i)$ is convex and increasing in x_i.*

Proof of Proposition 9 We first prove that $U_i(x_i)$ is convex. Fix $x \in X$ and $m_i \in M_i$. Observe that receiver i's conditional expected payoff, scaled by the probability of observing message m_i,[19] can be obtained by

$$U_i(x_i|m_i) = \max_{a_i \in A_i} \int_{\text{supp}(F_i)} u_i(\theta_i, a_i)\Big(x_i[(g_i(m_i|\theta_i) - \underline{g}_i(m_i)] + \underline{g}_i(m_i)\Big)dF(\theta_i)$$

Observe that $U_i(x_i|m_i)$ is the point-wise maximum of a family of linear functions of x_i, and is therefore convex in x_i. From here, since $U_i(x_i) = \int_{m_i \in M_i} U_i(x_i|m_i)dm_i$, and integration preserves convexity, $U_i(x_i)$ is convex in x_i. The second claim fol-

[18]It is straightforward to extend the analysis to cases where M_i is finite and G_i a discrete distribution.

[19]This is as multiplying the receiver's conditional expected payoff by the probability of observing a message (which is a scalar from the receiver's perspective) does not change the set of maximizers for the receiver.

lows as for all x_i,

$$\begin{aligned}U_i(x_i|m_i) &= \int_{m_i\in M_i}\left(\max_{a_i\in A_i}\int_{\text{supp}(F_i)} u_i(\theta_i,a_i)\Big(x_i[(g_i(m_i|\theta_i)-g_i(m_i)]+g_i(m_i)\Big)dF(\theta_i)\right)dm_i\\ &\geq \max_{a_i\in A_i}\int_{m_i\in M_i}\left(\int_{\text{supp}(F_i)} u_i(\theta_i,a_i)\Big(x_i[(g_i(m_i|\theta_i)-g_i(m_i)]+g_i(m_i)\Big)dF(\theta_i)\right)dm_i\\ &= \max_{a_i\in A_i}\int_{\text{supp}(F_i)} u_i(\theta_i,a_i)dF_i(\theta_i) = U_i(0)\end{aligned}$$

This completes our proof. $\square$

A2: Additional details for comparative statics

In this section, we provide a self-contained introduction to the relevant monotone comparative statics machinery required in this paper. This section serves both to complement the brief introduction to required terminology in Section 2.2., and to introduce concepts relevant to the proofs of Appendix B.

Let $f:\tilde{X}\to\mathbb{R}$, where $(\tilde{X},\succsim)$ is a partially-ordered set (henceforth POSET). We say that f is *increasing* in x if for all $x,x'\in\tilde{X}$, $x\succsim x'$ implies $f(x)\geq f(x')$. Now, suppose that $(\tilde{X},\succsim)$ and $(\Omega,\triangleright)$ are POSETS and $f:\tilde{X}\times\Omega\to\mathbb{R}$. Then, f displays *increasing differences* (*decreasing differences*) with respect to (x,t) if for all $x'\succsim x$ and $\omega'\triangleright\omega$, $f(x',\omega')-f(x,\omega')\geq f(x',\omega)-f(x,\omega)$ $(f(x',\omega')-f(x,\omega')\leq f(x',\omega)-f(x,\omega))$.

In our environment, $f(x,\omega)$ corresponds to $V(x,\omega)$: Sender's payoff given allocation x and parameter of concern ω. Additionally, $\tilde{X}$ corresponds to either of the sets $X'=\{x\in X:x_1\geq x_2\geq\cdots\geq x_N\}$ or $\overline{X}'=\{x\in\overline{X}:x_1\geq x_2\geq\cdots\geq x_N\}$, i.e., the elements of X or $\overline{X}$ with coordinates arranged in decreasing order. Both sets are partially ordered by $\succ^M$ and $\succ^M_w$. Meanwhile, $\Omega\subset\mathbb{R}$ is often an interval and always ordered by $\geq$, the usual ordering on $\mathbb{R}$. Thus, we refrain from explicitly mentioning the underlying partial order on Ω moving forward.

Now, $V(x,\omega)$ displays increasing differences in (x,ω), where the order on x is $\succ^M$, if for all $\omega\geq\omega'$, $V(x,\omega)-V(x,\omega')$ is increasing in the majorization order. Functions with domain X' or $\overline{X}'$ which are increasing (decreasing) with respect to the majorization order are called *schur-convex* (*schur-concave*) functions, i.e., $x \succ^M y$ implies $f(x) \geq f(y)$. If $f(x)$ is symmetric in each coordinate and convex (concave), then it is schur-convex (schur-concave). Additionally, when $f(x) = \sum_{i=1}^N g(x_i)$, $g : \mathbb{R} \to \mathbb{R}$, a sufficient condition for f to be schur-convex (schur-concave) in x is for $g(x_i)$ to be convex (concave) in x_i.[20] Thus, when $V(x, \omega) = \sum_{i=1}^N U(x_i, \omega)$, to show that $V(x, \omega) - V(x, \omega')$ is increasing (decreasing) in the majorization order on x, one need only show that $U(x_i, \omega) - U(x_i, \omega')$ is convex (concave) in x_i. When $V(x, \omega)$ is continuously differentiable in (x, ω), showing that $V(x, \omega)$ displays increasing differences (decreasing differences) in (x, ω), with respect to $\succ^M$, reduces to showing $\frac{\partial^3 U}{\partial \omega \partial x_i^2} \geq 0$ ($\frac{\partial^3 U}{\partial \omega \partial x_i^2} \leq 0$).

In the case of the weak majorization order $\succ_w^M$, a related condition applies. $V(x, \omega) - V(x, \omega')$ is increasing (decreasing) in x whenever $U(x_i, \omega) - U(x_i, \omega')$ is increasing and convex (concave) in x_i. When $V(x, \omega)$ is continuously differentiable in (x, ω), showing that $V(x, \omega)$ displays increasing differences (decreasing differences) in (x, ω), with respect to $\succ^M$, reduces to showing $\frac{\partial^2 U}{\partial \omega \partial x_i} \geq 0$ and $\frac{\partial^3 U}{\partial \omega \partial x_i^2} \geq 0$ ($\frac{\partial^3 U}{\partial \omega \partial x_i^2} \leq 0$).

Ideally, we wish to identify how the set of maximizers, $\arg\max_{x \in \overline{X}} V(x, \omega) = \overline{X}^*(\omega)$ or $\arg\max_{x \in X} V(x, \omega) = X^*(\omega)$ changes with respect to a change in the parameter ω. For instance, if $V(x, \omega)$ displays increasing differences in (x, ω) (where the order on x is $\succ^M$), then Milgrom & Shannon's (1994) celebrated monotonicity theorem (1994) implies that sufficient conditions for $\overline{X}^*(\omega)$ to be increasing in the strong set order with respect to $\succ^M$, is for $V(x, \omega)$ to be quasisupermodular in x for all ω, and for $(X, \succ^M)$ to be a lattice. While the latter is well known to be true, showing that $V(x, \omega)$ is quasisupermodular is often difficult when the underlying order is the majorization or weak-majorization order. Thus, we instead utilize the weaker notion of "non-decreasing" and "non-increasing", introduced in Section 2.2, throughout our piece.[21]

[20] Refer to Marshall et al. (1979) for additional concepts regarding majorization and schur-convexity / schur-concavity.

[21] A notable exception is when $N = 2$, and one considers the majorization order and restricts attention to elements of $\overline{X}$. If so, then all elements are ordered by the majorization order such that $V(x, \omega)$ is modular (and so quasisupermodular) on $\overline{X}$.

Next, the following result provides a connection between the concepts of non-decreasing and non-increasing, and increasing differences and decreasing differences respectively in the case where $(\tilde{X}, \succsim)$ is a POSET.

Lemma 3. *Let $(\tilde{X}, \succsim)$ and $(\Omega, \trianglerighteq)$ be POSETS, $f : X \times \Omega \to \mathbb{R}$ and f displays increasing (decreasing) differences in (x, ω). Then, $\tilde{X}^*(\omega) = \arg\max_{x \in \tilde{X}} V(x, \omega)$ is non-decreasing (non-increasing) in ω.*

Proof We prove the claim when f displays increasing differences in (x, ω), noting that the proof for decreasing differences follows an identical format. Take $\omega \trianglerighteq \omega'$, and take $x \in \tilde{X}^*(\omega)$ and $x' \in \tilde{X}^*(\omega')$ which are comparable under $\succsim$. Without loss, we assume that either $x \succ x'$ or $x' \succ x$ holds. Evidently, the claim is automatically fulfilled if $x = \max\{x, x'\}$, which implies $x' = \min\{x, x'\}$. Hence, suppose $x = \min\{x, x'\}$, which implies $x' = \max\{x, x'\}$. Then, since $0 \le f(x, \omega) - f(x', \omega) \le f(x, \omega') - f(x', \omega')$, we must have $x \in \tilde{X}^*(\omega')$. Reversing the inequality further implies $x' \in \tilde{X}^*(\omega)$, which proves the claim. $\square$

To understand where the lemma applies, suppose that $V(x, \omega)$ displays increasing differences in (x, ω) with respect to $\succ^M$ on $\overline{X}$. Label $(\overline{X}^*)'(\omega) = \arg\max_{x \in \overline{X}'} V(x, \omega)$. Since $(\overline{X}', \succ^M)$ and $(\Omega, \ge)$ are POSETs, $(\overline{X}^*)'(\omega)$ is non-decreasing in ω. Now, define $\overline{X}^*(\omega) = \arg\max_{x \in \overline{X}} V(x, \omega)$. Take any $\omega, \omega' \in \Omega$ under which $\omega > \omega'$, and take any $\succ^M$-comparable $x \in \overline{X}^*(\omega)$ and $x' \in \overline{X}^*(\omega')$. By symmetry of $V(x, \omega)$, $[x] \in (\overline{X}^*)'(\omega)$ and $[x'] \in (\overline{X}^*)'(\omega')$. By the definition of non-decreasing, we have $\max\{[x], [x']\} \in (\overline{X}^*)'(\omega)$ and $\min\{[x], [x']\} \in (\overline{X}^*)'(\omega')$, which implies $\max\{[x], [x']\} \in \overline{X}^*(\omega)$ and $\min\{[x], [x']\} \in \overline{X}^*(\omega')$. Hence, $\overline{X}^*(\omega)$ is non-decreasing with respect to $\succ^M$ in ω. Reversing the logic finds that $V(x, \omega)$ displays decreasing differences in (x, ω) with respect to $\succ^M$ on $\overline{X}$ implies $\overline{X}^*(\omega)$ is non-increasing with respect to $\succ^M$ in ω. One may also extend these concepts appropriately to $X^*(\omega) = \arg\max_{x \in X} V(x, \omega)$ being non-decreasing and non-increasing in ω with respect to $\succ^M$ and $\succ^M_w$.

Combining these with the prior discussion on sufficient conditions for schur-convexity and schur-concavity yields the following

Remark 1. *Suppose $V(x,\omega)$ is differentiable in (x,ω)*

- *If $\frac{\partial V(x,\omega)}{\partial \omega}$ is schur-convex (schur-concave) in x, then $\overline{X}^*(\omega) = \arg\max_{x\in\overline{X}} V(x,\omega)$ is non-decreasing (non-increasing) with respect to $\succ^M$ in ω. Sufficient conditions for this include (i) $\frac{\partial V(x,\omega)}{\partial \omega}$ being symmetric and convex (symmetric and concave) in x, and (ii) when $V(x,\omega) = \sum_{i=1}^N U(x_i,\omega)$ and $\frac{\partial V(x,\omega)}{\partial \omega}$ is twice-differentiable in x_i, $\frac{\partial^3 U}{\partial \omega \partial x_i^2} \geq 0$ ($\frac{\partial^3 U}{\partial \omega \partial x_i^2} \leq 0$)*

- *If $\frac{\partial V(x,\omega)}{\partial \omega}$ is schur-convex (schur-concave) and increasing (decreasing) in x, then $X^*(\omega) = \arg\max_{x\in X} V(x,\omega)$ is non-decreasing (non-increasing) with respect to $\succ_w^M$ in ω. Sufficient conditions for this include (i) $\frac{\partial V(x,\omega)}{\partial \omega}$ being symmetric, convex and increasing (symmetric, concave and decreasing) in x, and (ii) when $V(x,\omega) = \sum_{i=1}^N U(x_i,\omega)$ and $\frac{\partial V(x,\omega)}{\partial \omega}$ is twice-differentiable in x_i, $\frac{\partial^2 U}{\partial \omega \partial x_i} \geq 0$ and $\frac{\partial^3 U}{\partial \omega \partial x_i^2} \geq 0$ ($\frac{\partial^3 U}{\partial \omega \partial x_i^2} \leq 0$).*

We apply Remark (1) throughout the proofs of Appendix B.

B: Proofs

Proof of Proposition 1 See Appendix A1. □

Proof of Corollary 1 Since $U(x_i)$ is increasing in x_i, it is without loss to focus on $x \in \overline{X}$. By Proposition 1, $U(x_i)$ is convex in x_i. Hence, $V(x) = \sum_{i=1}^N U(x)$ is convex in x, and maximized at either of the extreme points of $\overline{X}$, i.e., by fully targeting a single receiver, by Bauer's maximum theorem. □

Proof of Corollary 2 We prove the claim when $C_0 > 0$, noting that the proof is easily extended to when $C_0 \leq 0$.

Let $\mathbb{E}[\theta|m, x_i]$ denote the posterior mean of a representative receiver's state given message observed m and allocation x_i. We note that

$$\mathbb{E}[\theta|m, x_i] = \int_{\text{supp}(F)} \theta\bigg(\frac{x_i g(m|\theta) + (1-x_i)g(m)}{g(m)}\bigg)dF(\theta)$$
$$= x_i \int_{\text{supp}(F)} \theta\bigg(\frac{g(m|\theta)}{g(m)}\bigg)dF(\theta) + (1-x_i)\int_{\text{supp}(F)} \theta\bigg(\frac{g(m)}{g(m)}\bigg)dF(\theta)$$
$$= x_i\mathbb{E}[\theta|m,1] + (1-x_i)\mathbb{E}[\theta] \tag{17}$$

Hence, by corollary's assumptions, $a^*_m(x_i) = x_i C_0[\mathbb{E}[\theta|m,1] + (1-x_i)\mathbb{E}[\theta]] - C_1$. Thus, for all m such that $\mathbb{E}[\theta|m,1] \leq \mathbb{E}[\theta]$, $a^*_m(x_i)$ is decreasing in x_i. Likewise, for all m such that $\mathbb{E}[\theta|m,1] \geq \mathbb{E}[\theta]$, $a^*_m(x_i)$ is increasing in x_i. Further label the default action as $a^*_m(0) = a^D$. Finally, let

$$\underline{a} = C_0 \min_{m\in M} \mathbb{E}[\theta|m,1] + C_1, \quad \overline{a} = C_0 \max_{m\in M} \mathbb{E}[\theta|m,1] + C_1$$

Where we note that for all $m \in M$ and $x_i \in [0,1]$, $a^*_m(x_i) \in [\underline{a}, \overline{a}]$. Now, for any $x \in [0,1]$,

$$\int_{\underline{a}}^{\overline{a}} a dH(a|x) = \int_0^1 a^*_m(x_i)g(m)dm$$
$$= x_i C_0\bigg(\int_0^1 \mathbb{E}[\theta|m] - \mathbb{E}[\theta]g(m)dm\bigg) + \int_0^1 (\mathbb{E}[\theta] + C_1)g(m)dm$$
$$= (\mathbb{E}[\theta] + C_1)\int_0^1 g(m)dm = (\mathbb{E}[\theta] + C_1)$$

Where the second inequality holds as $\int_0^1 \mathbb{E}[\theta|m,1]g(m)dm = \mathbb{E}[\theta]$ via the law of iterated expectations. Hence, for any $x_i \in [0,1]$, $H(a|x_i)$ has the same mean. Next, take any pair x_i, x'_i which satisfies $0 \leq x'_i < x_i \leq 1$. Take any $a \in [\underline{a}, \overline{a}]$ under which $a \leq a^D$. By the discussion above, $a^*_m(x'_i) \leq a$ implies $a^*_m(x_i) \leq a$. Hence, $\{m : a^*_m(x'_i) \leq a\} \subseteq \{m : a^*_m(x_i) \leq a\}$, and so $H(a|x'_i) \leq H(a|x_i)$. Thus, for any such a, we have $\int_{\underline{a}}^{a} H(a'|x'_i)da' \leq \int_{\underline{a}}^{a} H(a'|x_i)da'$. Meanwhile, fix any $a > a^D$. By a similar logic, $a^*_m(x'_i) > a$ implies $a^*_m(x_i) > a$. Hence, $\{m : a^*_m(x'_i) > a\} \subseteq \{m : a^*_m(x_i) > a\}$. Noting that $1 - H(a|\tilde{x}_i) = \int_{\{m:a^*_m(\tilde{x}_i)>a\}} g(m)dm$, we therefore have $1-$

$H(a|x_i') \leq 1 - H(a|x_i)$, or $H(a|x_i') \geq H(a|x_i)$. Thus, $\int_a^{\overline{a}} H(a'|x_i')da' \geq \int_a^{\overline{a}} H(a'|x_i')da'$, which implies $\int_{\underline{a}}^{a} H(a'|x_i')da' \leq \int_{\underline{a}}^{a} H(a'|x_i')da'$ since both distributions have the same mean. Hence, $H(a|x_i)$ is a mean-preserving spread of $H(a|x_i')$. $\square$

Proof of Proposition 2 Immediately follows from the observation that $U(x_i) = U_L(\varepsilon(x_i))$, where $\varepsilon(x_i)$ is convex, and $U_L(x_i)$ by Proposition 1 is convex and increasing in x_i. $\square$

Proof of Proposition 3 We consider each possibility separately

Possibility 1: $\varepsilon(x_i) = x_i^{1-\rho}$. First, observe that $V(x)$ has the same set of maximizers as $\frac{1}{1-\rho}V(x)$. Hence, we work with the latter, and pre-multiply all receivers' payoffs by $\frac{1}{1-\rho}$. Doing so, twice differentiating a receiver's payoff by x_i yields

$$U''(x_i) = \underbrace{-\rho x_i^{-\rho-1} U_L'(x_i^{1-\rho})}_{(i)} + \underbrace{(1-\rho) x_i^{-2\rho} U_L''(x_i^{1-\rho})}_{(ii)}$$

We will show that the magnitude of term (i) (which is negative) is increasing in ρ, while the magnitude of term (ii) (which is positive) is decreasing in ρ. Together, these imply that $\frac{\partial U''(x_i)}{\partial \rho} = \frac{\partial^3 U(x_i)}{\partial^2 x_i \partial \rho} \leq 0$. Remark 1 then implies $\overline{X}^*(\rho)$ is non-increasing with respect to $\succ^M$ in ρ.

We begin with (i). We note that $\frac{\partial}{\partial \rho}(\rho x_i^{-\rho-1}) = x_i^{-\rho-1} + -\rho \log(x_i) x_i^{-\rho-1} > 0$. Meanwhile, since $x_i \in [0,1]$, $x_i^{1-\rho}$ is increasing in ρ, while $U_L'(x_i)$ is strictly increasing in x_i by the strict convexity of $U_L(x_i)$. Thus, $U_L'(x_i^{1-\rho})$ is increasing in ρ. That both terms are positive then implies the magnitude of (i) is increasing in ρ.

We now consider (ii). We first show that $(1-\rho)x_i^{-2\rho}$ is always weakly decreasing in ρ. To see this, observe

$$\frac{\partial}{\partial \rho}((1-\rho)x_i^{-2\rho}) = x_i^{-2\rho}[-2(1-\rho)\log(x_i) - 1] := k(x_i, \rho)$$

We note

$$\frac{\partial k(x_i, \rho)}{\partial x_i} = 2\rho x^{-2\rho-1}[2(1-\rho)\log(x_i) + 1]$$

Letting x^R (uniquely) solve $[2(1-\rho)\log(x^R)+1] = 0$, we find that $\frac{\partial k(x_i,\rho)}{\partial x_i} > 0$ for all $x_i < x^R$, $\frac{\partial k(x_i,\rho)}{\partial x_i} < 0$ for all $x_i > x^R$, while $\frac{\partial k(x_i,\rho)}{\partial x_i} = 0$ for $x_i = x^R$. This implies that $k(x_i, \rho)$ is concave and maximized when $2(1-\rho)\log(x_i) + 1]$ holds, under which it is equal to zero. Thus, $(1-\rho)x_i^{-2\rho}$ is always weakly decreasing in ρ. Meanwhile, since $U_L'''(x_i) \leq 0$, $U_L''(x_i^{1-\rho})$ is decreasing in ρ. That both terms are positive then implies (ii) is decreasing in ρ. Hence, $\overline{X}^*(\rho)$ is non-increasing with respect to $\succ^M$ in ρ.

Possibility 2: Now suppose $\varepsilon(x_i) = \rho + (1-\rho)\tilde{\varepsilon}(x_i)$, where $\tilde{\varepsilon}(x_i)$ possesses the properties stated in the statement of the proposition. Once again, we pre-multiply all receivers' payoffs by $\frac{1}{1-\rho}$. Doing so, twice differentiating a receiver's payoff by x_i yields

$$U''(x_i) = \underbrace{\varepsilon''(x_i)U_L'(\rho + (1-\rho)\varepsilon(x_i))}_{(i)} + \underbrace{(1-\rho)[\varepsilon(x_i)]^2 U_L''(\rho + (1-\rho)\varepsilon(x_i))}_{(ii)}$$

Since $1 - \varepsilon(x_i) \geq 0$ for any $x_i \in [0,1]$, $\rho + (1-\rho)\varepsilon(x_i)$ is increasing in ρ, which implies the magnitude of (i) (which is negative) is increasing in ρ. Meanwhile, $(1-\rho)$ is decreasing in ρ, while $U_L''(\rho + (1-\rho)\varepsilon(x_i))$ is also decreasing in ρ. Hence, the magnitude of (ii) (which is positive) is decreasing in ρ. From here, applying a similar argument to possibility 1 yields that $\overline{X}^*(\rho)$ is non-increasing with respect to $\succ^M$ in ρ. $\square$

Proof of Corollary 3 First, suppose $(\varepsilon(x_i))^2$ is convex in x_i. Then, $V(x) = \sum_{i=1}^N U(x)$ is convex in x, and maximized at either of the extreme points of $\overline{X}$, i.e., by fully targeting a single receiver, by Bauer's maximum theorem. Next, suppose $(\varepsilon(x_i))^2$ is concave in x_i. Then, $V(x)$ is schur-concave in x and thus maximized at the $\succ^M$-minimal allocation in $\overline{X}$, $(1/N, ..., 1/N)$. $\square$

Proof of Proposition 4 Since $U(x_i)$ is strictly convex on x_i, it must be strictly increasing on X. Hence, without loss, we focus on Sender's choices on $\overline{X}$. Label the fully targeted allocation which sets $x_i = 1$ as δ^i, and label the equal spread allocation $x^B = (1/N, 1/N, ..., 1/N)$, i.e., the fully equal allocation. We first show that for all $x \notin \{\delta^i\}_{i=1}^N \cup \{x^B\}$, $\max\{V(\delta^1), ..., V(\delta^N), V(x^B)\} > V(x)$. Take any such x, and any $i \in \arg\min_j U(x_j)$. If so, then $x \in \overline{X}_i := \{x' : \min_{j \in \mathcal{N}} U(x'_j) = U(x'_i)\}$. Furthermore, Sender's payoff on $\overline{X}_i$ is given by

$$V(x) = \frac{\beta}{N} \sum_{i=1}^{N} U(x_i) + (1-\beta) U(x_i)$$

As $U(x_i)$ is strictly convex in x_i, $V_i(x)$ is strictly convex on $\overline{X}_i$ for each i. Furthermore, $\overline{X}_i$ is a non-empty closed, compact convex set with extreme points being all the vertices of $\overline{X}$, excluding δ^i, and the equal allocation $(1/N, 1/N, ..., 1/N) := x^B$. This implies $\max_{j \neq i}\{V(\delta^j), V(x^B)\} = \max_{x' \in \overline{X}_i} V(x')$. It also implies that there exists a collection of non-negative scalars $\{\lambda_j\}_{j=1}^N$, $\sum_{j=1}^N \lambda_j = 1$, such that $x = \sum_{j \neq i} \lambda_j \delta^j + \lambda_i x^B$. Combined, we obtain

$$\max_{j \neq i}\{V(\delta^j), V(x^B)\} \geq \sum_{j \neq i} \lambda_j V(\delta^j) + \lambda_i V(x^B) > V(\sum_{j \neq i} \lambda_j \delta^j + \lambda_i x^B) = V(x)$$

That x was arbitrary proves the claim.

Thus, for any β, one need only compare Sender's payoff under δ^i for some i, and that under $(1/N, 1/N, ..., 1/N)$. For any i, $V(\delta^i) = \frac{1}{N}(\beta(U(1) - U(0)) + U(0))$ while $V(x^B) = U(\frac{1}{N})$. Defining $\tilde{V}(\beta) = N(V(\delta^i) - V(x^B))$, observe

$$\tilde{V}(\beta) = \beta(U(1) - U(0)) - N(U(\frac{1}{N}) - U(0))$$

Which is strictly increasing and continuous in β (given $U(1) - U(0) > 0$). From here, $\tilde{V}(\overline{\beta}) = 0$, and for all $\beta > \overline{\beta}$ ($\beta < \overline{\beta}(N)$), $\tilde{V}(\beta) > 0$ ($\tilde{V}(\beta) < 0$) such that fully-targeting (equal allocation) is strictly optimal for Sender. □

Proof of Corollary 4 To see that $\overline{\beta}(N)$ is strictly increasing in N, we need only show that $y[U(\frac{1}{y}) - U(0)]$ is strictly decreasing in y on $y \geq 1$. To see why this holds, observe that the equation of the tangent to $U(x_i)$ at $x_i = \frac{1}{y}$ is given by $\frac{1}{y}U'(\frac{1}{y}) + K$

for some $K \in \mathbb{R}$. Because U is strictly convex and increasing on $[0,1]$, for all $y \geq 1$, $K < U(0)$. Hence,

$$\frac{\partial}{\partial y}(y[U(\frac{1}{y}) - U(0)]) = U(\frac{1}{y}) - [\frac{1}{y}U'(\frac{1}{y}) + U(0)] < U(\frac{1}{y}) - [\frac{1}{y}U'(\frac{1}{y}) + K] = 0$$

Which proves the claim. $\square$

Proof of Proposition 5 To begin, we note that under the quadratic loss form, the receiver's optimal action is simply the posterior mean, which is given by $a^*_m(x_i) = x_i\mathbb{E}[\theta|1,m] + (1-x_i)\mathbb{E}[\theta]$ as per (17). First, let $\beta = 0$. If so, then $V(x,0) = \sum_{i=1}^N \int_0^1 w(x_i\mathbb{E}[\theta|1,m] + (1-x_i)\mathbb{E}[\theta])g(m)dm$. Observe that $\frac{\partial^2 V(x,0)}{\partial x_i \partial x_j} = 0$ for $i \neq j$, while for each i,

$$\frac{\partial^2 V(x,0)}{\partial x_i^2} = \int_0^1 \bigg([\mathbb{E}[\theta|m,1] - \mathbb{E}[\theta]]^2 w''(a^*_m(x_i)) \bigg) g(m)dm$$

Notice that if w is convex, then for each $m \in M$, $[\mathbb{E}[\theta|m,1] - \mathbb{E}[\theta]]^2 w''(a^*_m(x_i)) \geq 0$. Hence, $\frac{\partial^2 V(x,0)}{\partial x_i^2} \geq 0$. That $V(x,0)$ is symmetric in x then implies it is schur-convex in x. Meanwhile, if w is concave, then for each $m \in M$, $[\mathbb{E}[\theta|m,1] - \mathbb{E}[\theta]]^2 w''(a^*_m(x_i)) \leq 0$. Hence, $\frac{\partial^2 V(x,0)}{\partial x_i^2} \leq 0$, which implies by a similar reasoning to the above that $V(x,0)$ is schur-concave in x.

Now, suppose w is convex. Observe that $V(x,\beta) = \beta V(x,1) + (1-\beta)V(x,0)$, which is the sum of schur-convex functions in x, and is thus schur-convex in x. We first show that there exists an optimal allocation which lies in $\overline{X}$. Take any $x \in X$ such that $\sum_{i=1}^N x_i < 1$. Then, there exists i such that $x_i < 1$. Consider the allocation x' where $x'_i = x_i + \varepsilon$ for small $\varepsilon > 0$, and $x'_j = x_j$ for $j \neq i$. Observe that

$$\begin{aligned} V(x',\beta) - V(x|\beta) &= \beta[U(x_i+\varepsilon) - U(x_i)] \\ &\quad + (1-\beta)[\int_0^1 w(a^*_m(x_i+\varepsilon))g(m)dm - \int_0^1 w(a^*_m(x_i))g(m)dm] \\ &\geq (1-\beta)[\int_{\underline{a}}^{\overline{a}} w(a)dH(a|x_i+\varepsilon) - \int_{\underline{a}}^{\overline{a}} w(a)dH(a|x_i)] \geq 0 \end{aligned}$$

Where the first inequality holds as $U(x_i)$ is increasing in x_i, and the equivalence of $\int_0^1 w(a_m^*(x_i))g(m)dm$ and $\int_{\underline{a}}^{\overline{a}} w(a)dH(a|x_i)$ by the definition of $H(a|x_i)$ from (4), and the last inequality holds as $H(a|x_i+\varepsilon)$ is a mean-preserving spread of $H(a|x_i)$ by Corollary 2, and $w(\cdot)$ is convex in a. Hence, it is without loss to focus on maximizing $V(x,\beta)$ on $\overline{X}$. That $V(x,\beta)$ is schur-convex then implies that fully targeting a single receiver is optimal for Sender.

Next, suppose that w is concave. We notice that $V(x,\beta)$ has the same set of maximizers as $\tilde{V}(x) = \frac{1}{\beta}V(x,\beta)$, i.e., $\max_{x\in X}\tilde{V}(x) = X^*(\beta)$. Now, let $\gamma = \frac{\beta-1}{\beta}$, and observe that $\frac{\partial \tilde{V}(x)}{\partial \gamma} = -V(x,0)$,which is schur-convex in x. Furthermore, since

$$-V(x,0) = \sum_{i=1}^{N}\int_{\underline{a}}^{\overline{a}}[-w(a)]dH(a|x_i)$$

and $-w(a)$ is convex, Corollary 2 implies that $-V(x|0)$ is increasing in x on X. Combined with Remark (1) and the fact that γ is strictly increasing in β, $X^*(\beta)$ is non-decreasing with respect to $\succ_w^M$ in β. $\square$

Proof of Proposition 6 We first identify the equilibrium actions of each receiver. Take any $\boldsymbol{m}\in M^2$. Label $\Delta_{\boldsymbol{m}}(x) = a^*_{\boldsymbol{m},2}(x) - a^*_{\boldsymbol{m},1}(x)$. By each receiver's first-order condition, we note that $a^*_{\boldsymbol{m},1}(x)$ and $a^*_{\boldsymbol{m},2}(x)$ must satisfy

$$a^*_{\boldsymbol{m},1}(x) = \mathbb{E}[\theta_1|m_1,x_1] + \phi s'(\Delta_{\boldsymbol{m}}(x)), \quad a^*_{\boldsymbol{m},2}(x) = \mathbb{E}[\theta_2|m_2,x_2] + \phi s'(-\Delta_{\boldsymbol{m}}(x))$$

Where $\mathbb{E}[\theta_i|m_i|x_i] = \int_F (\theta_i \frac{x_i g(m_i|\theta_i)+(1-x_i)g(m_i)}{g(m_i)})dF(\theta_i)$. Note that by symmetry of $s(\cdot)$ about 0, $s'(\Delta_{\boldsymbol{m}}(x)) = -s'(-\Delta_{\boldsymbol{m}}(x))$. Thus, subtracting the first equation from the second yields

$$\Delta_{\boldsymbol{m}}(x) = \mathbb{E}[\theta_2|m_2,x_2] - \mathbb{E}[\theta_1|m_1,x_1] - 2\phi s'(\Delta_{\boldsymbol{m}}(x)) \tag{18}$$

Through implicit differentiation, we obtain

$$\frac{\partial \Delta_m(x)}{\partial x_1} = \frac{\mathbb{E}[\theta] - \mathbb{E}[\theta_1|m_1, 1]}{1 + 2\phi s''(\Delta_m(x))}, \quad \frac{\partial \Delta_m(x)}{\partial x_2} = \frac{\mathbb{E}[\theta_2|m_2, 2] - \mathbb{E}[\theta]}{1 + 2\phi s''(\Delta_m(x))}$$

$$\frac{\partial^2 \Delta_m(x)}{\partial x_1 \partial x_2} = \left(\frac{\partial \Delta_m(x)}{\partial x_1}\right)\left(\frac{\partial \Delta_m(x)}{\partial x_2}\right)\left(\frac{-2\phi s'''(\Delta_m(x))}{1 + 2\phi s''(\Delta_m(x))}\right)$$

$$\frac{\partial^2 \Delta_m(x)}{\partial x_i^2} = (\frac{\partial \Delta_m(x)}{\partial x_i})^2 \frac{-2\phi s'''(\Delta_m(x))}{1 + 2\phi s''(\Delta_m(x))}$$

Meanwhile, adding the first-order conditions for both receivers yields

$$a^*_{\boldsymbol{m},1}(x) + a^*_{\boldsymbol{m},2}(x) = \mathbb{E}[\theta_1|m_1, x] + \mathbb{E}[\theta_2|m_2, 1 - x]$$

Which implies $\frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} = -\frac{\partial a^*_{\boldsymbol{m},2}(x)}{\partial \phi}$. Differentiating receiver 1's first-order condition with respect to ϕ then yields

$$\begin{aligned}
\frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} &= s'(\Delta_m(x)) + \phi s''(\Delta_m(x))[\frac{\partial a^*_{\boldsymbol{m},2}(x)}{\partial \phi} - \frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi}] \\
&= s'(\Delta_m(x)) - 2\phi s''(\Delta_m(x))\frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} \\
\iff \frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} &= \frac{s'(\Delta_m(x))}{1 + 2\phi s''(\Delta_m(x))}
\end{aligned}$$

Now, let $U_{\boldsymbol{m},1}(x)$ and $U_{\boldsymbol{m},2}(x)$ denote receiver 1 and 2's payoffs conditional on message $\boldsymbol{m}$ and allocation x. Applying the envelope theorem, we note that

$$\begin{aligned}
\frac{\partial U_{\boldsymbol{m},1}(x)}{\partial \phi} &= -2s(\Delta_{\boldsymbol{m}}(x)) - 2\phi \frac{\partial a^*_{\boldsymbol{m},2}(x)}{\partial \phi} s'(\Delta_{\boldsymbol{m}}(x)) \\
&= -2s(\Delta_{\boldsymbol{m}}(x)) + 2\phi \frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} s'(\Delta_{\boldsymbol{m}}(x)) \\
\frac{\partial U_{\boldsymbol{m},2}(x)}{\partial \phi} &= -2s(-\Delta_{\boldsymbol{m}}(x)) - 2\phi \frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} s'(-\Delta_{\boldsymbol{m}}(x)) \\
&= -2s(\Delta_{\boldsymbol{m}}(x)) + 2\phi \frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} s'(\Delta_{\boldsymbol{m}}(x))
\end{aligned}$$

Hence, the Sender's conditional expected payoff on message $\boldsymbol{m}$, differentiated with respect to ϕ, as

$$\frac{\partial V_{\boldsymbol{m}}(x,\phi)}{\partial\phi} = \frac{\partial U_{\boldsymbol{m},1}(x)}{\partial\phi} + \frac{\partial U_{\boldsymbol{m},2}(x)}{\partial\phi} = -4\bigg(s(\Delta_{\boldsymbol{m}}(x)) - \phi\frac{(s'(\Delta_{\boldsymbol{m}}(x)))^2}{(1+2\phi s''(\Delta_{\boldsymbol{m}}(x)))}\bigg)$$

Let us label $h(\Delta) = s(\Delta) - \phi\frac{(s'(\Delta))^2}{(1+2\phi s''(\Delta))}$, and label the determinants of the first and second principal minors of the hessian of $\frac{\partial V_{\boldsymbol{m}}(x,\phi)}{\partial\phi}$ as $det(H_1)$ and $det(H_2)$ (with x_i for any $i = 1,2$ in the first entry) We then obtain

$$\begin{aligned}
det(H_1) &= \frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial\phi\partial x_i^2} = -4\bigg(\frac{\partial^2\Delta_m(x)}{\partial x_i^2}h'(\Delta_m(x)) + h''(\Delta_m(x))\bigg[\frac{\partial\Delta_m(x)}{\partial x_i}\bigg]^2\bigg) \\
&= -4\bigg[\frac{\partial\Delta_m(x)}{\partial x_i}\bigg]^2 \underbrace{\bigg(\frac{-2\phi s'''(\Delta_{\boldsymbol{m}}(x))}{1+2\phi s''(\Delta_{\boldsymbol{m}}(x))}h'(\Delta_m(x)) + h''(\Delta_m(x))\bigg)}_{\text{Simplifies to (12)}} \le 0
\end{aligned}$$

$$\begin{aligned}
det(H_2) &= \bigg(\frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial\phi\partial x_1^2}\bigg)\bigg(\frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial\phi\partial x_2^2}\bigg) - \bigg(\frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial\phi\partial x_1\partial x_2}\bigg)^2 \\
&= 4\bigg[\frac{\partial\Delta_m(x)}{\partial x_1}\bigg]^2\bigg(\frac{-2\phi s'''(\Delta_{\boldsymbol{m}}(x))}{1+2\phi s''(\Delta_{\boldsymbol{m}}(x))}h'(\Delta_m(x)) + h''(\Delta_m(x))\bigg)\times \\
&\quad \bigg[\frac{\partial\Delta_m(x)}{\partial x_2}\bigg]^2\bigg(\frac{-2\phi s'''(\Delta_{\boldsymbol{m}}(x))}{1+2\phi s''(\Delta_{\boldsymbol{m}}(x))}h'(\Delta_m(x)) + h''(\Delta_m(x))\bigg) \\
&\quad - 4\bigg(\bigg[\frac{\partial\Delta_m(x)}{\partial x_1}\frac{\partial\Delta_m(x)}{\partial x_2}\bigg]\bigg(\frac{-2\phi s'''(\Delta_{\boldsymbol{m}}(x))}{1+2\phi s''(\Delta_{\boldsymbol{m}}(x))}h'(\Delta_m(x)) + h''(\Delta_m(x))\bigg)\bigg)^2 = 0
\end{aligned}$$

Hence, the Hessian is negative semi-definite, which implies $\frac{\partial V_{\boldsymbol{m}}(x,\phi)}{\partial\phi}$ (for each message $\boldsymbol{m}$) is concave in x. This implies that $\frac{\partial V(x,\phi)}{\partial\phi} = \int_{\boldsymbol{m}}\frac{\partial V_{\boldsymbol{m}}(x,\phi)}{\partial\phi}g(\boldsymbol{m})d\boldsymbol{m}$ is concave in x. That it is a symmetric function implies it is schur-concave in x.

We now also show that $\frac{\partial V(x,\phi)}{\partial\phi}$ is decreasing in x. Let $T(\Delta|x)$ denote the distribution over $\Delta_m(x)$, which is defined on $(-1,1)$ and given by

$$T(\Delta|x) = \int_{\{\boldsymbol{m}:\Delta_{\boldsymbol{m}}(x)\le\Delta\}} g(\boldsymbol{m})d\boldsymbol{m}$$

Where $g(\boldsymbol{m}) = g(m_1)g(m_2)$. Furthermore, fix x_2 and $m_2 \in M$. Define

$$T_1(\Delta|x_1, x_2) = \int_{\{m_1 : \Delta_{(m_1,m_2)}(x_1,x_2) \leq \Delta\}} g(m_1)dm_1$$

Where $T(\Delta|x) = \int_{m_2} T_1(\Delta|x_1, x_2)g(m_2)dm_2$. Applying Fubini's theorem to interchange integrals, we also note that

$$\begin{aligned}\int_{-1}^{\Delta} T(\Delta'|x)d\Delta' &= \int_{-1}^{\Delta} \left(\int_{m_2} T_1(\Delta'|x_1, x_2)g(m_2)dm_2 \right) d\Delta' \\ &= \int_{m_2} \left(\int_{-1}^{\Delta} T_1(\Delta'|x_1, x_2)d\Delta' \right) g(m_2)dm_2 \end{aligned} \tag{19}$$

Now, take any $0 \leq x_1' \leq x_1 < 1$. Fixing any $x_2 \in [0, 1]$, our goal is to prove that $T(\Delta|x_1, x_2)$ is a mean-preserving spread of $T(\Delta|x_1', x_2)$. We break this proof into three steps.

Step 1: We prove that $\mathbb{E}[\Delta_{\boldsymbol{m}}(x)] = 0$ for all x. First, observe that by taking the unconditional expectation of (18), and rearranging:

$$\begin{aligned}&\Delta_{\boldsymbol{m}}(x) = \mathbb{E}[\theta_2|m_2, x_2] - \mathbb{E}[\theta_1|m_1, x_1] - 2\phi s'(\Delta_{\boldsymbol{m}}(x)) \\ &\Leftrightarrow \mathbb{E}[\Delta_{\boldsymbol{m}}(x)] + 2\phi\mathbb{E}[s'(\Delta_{\boldsymbol{m}}(x))] = \mathbb{E}[\mathbb{E}[\theta_2|m_2, x_2] - \mathbb{E}[\theta_1|m_1, x_1]] = 0\end{aligned} \tag{20}$$

Noting that s' is symmetric about 0, (20) implies

$$\mathbb{E}[-\Delta_{\boldsymbol{m}}(x)] + 2\phi\mathbb{E}[s'(-\Delta_{\boldsymbol{m}}(x))] = -\mathbb{E}[\mathbb{E}[\theta_2|m_2, x_2] + \mathbb{E}[\theta_1|m_1, x_1]] = 0 \tag{21}$$

By contradiction, first suppose $\mathbb{E}[\Delta_{\boldsymbol{m}}(x)] > 0$. Since s' is convex in x, we have $2\phi\mathbb{E}[s'(\Delta_{\boldsymbol{m}}(x))] \geq 2\phi s'(\mathbb{E}[\Delta_{\boldsymbol{m}}(x)]) > 0$, which implies $\mathbb{E}[\Delta_{\boldsymbol{m}}(x)] + 2\phi\mathbb{E}[s'(\Delta_{\boldsymbol{m}}(x))] > 0$; contradicting (20). Next, suppose $\mathbb{E}[\Delta_{\boldsymbol{m}}(x)] < 0$, i.e., $\mathbb{E}[-\Delta_{\boldsymbol{m}}(x)] > 0$. Convexity implies $2\phi\mathbb{E}[s'(-\Delta_{\boldsymbol{m}}(x))] \geq 2\phi s'(\mathbb{E}[-\Delta_{\boldsymbol{m}}(x)]) > 0$. This contradicts (21).

Step 2: Fix any $m_2 \in M$. Take any $\Delta \in (-1, 1)$ such that $\Delta \leq \mathbb{E}[\theta_2|m_2, x_2]$. Suppose, for some m_1, that $\Delta_{\boldsymbol{m}}(x_1', x_2) \leq \Delta$. Notice that the sign of $\frac{\partial \Delta_{\boldsymbol{m}}(x)}{\partial x_1}$ only depends on the sign of $\mathbb{E}[\theta] - \mathbb{E}[\theta|m_1]$, which must be weakly negative under m_1 as

$\Delta_{\boldsymbol{m}}(x_1', x_2) \leq \mathbb{E}[\theta_2|m_2, x_2]$. Hence, $\Delta_{\boldsymbol{m}}(x)$ is decreasing in x_1 for all $x_1 \in [0,1]$ given this message m_1, which implies $\Delta_{\boldsymbol{m}}(x_1, x_2) \leq \Delta$. As a result, we have for any such Δ, $\{m_1 : \Delta_{(m_1,m_2)}(x_1', x_2) \leq \Delta\} \subseteq \{m_1 : \Delta_{(m_1,m_2)}(x_1, x_2) \leq \Delta\}$, so $T_1(\Delta|x_1, x_2) \geq T_1(\Delta|x_1', x_2)$. In turn, $\int_{-1}^{\Delta} T_1(\Delta'|x_1, x_2)d\Delta' \geq \int_{-1}^{\Delta} T_1(\Delta'|x_1', x_2)d\Delta'$. Integrating over all m_2 and applying (19) yields $\int_{-1}^{\Delta} T(\Delta'|x_1, x_2)d\Delta' \geq \int_{-1}^{\Delta} T(\Delta'|x_1', x_2)d\Delta'$.

Step 3: Fix any $m_2 \in M$. Take any $\Delta \in (-1, 1)$ such that $\Delta \geq \mathbb{E}[\theta_2|m_2, x_2]$. Then, $\Delta_{\boldsymbol{m}}(x_1', x_2) \geq \Delta$ (which implies $\frac{\partial \Delta_{\boldsymbol{m}}(x)}{\partial x_1} \geq 0$ by a similar logic to Step 2) implies $\Delta_{\boldsymbol{m}}(x_1, x_2) \geq \Delta$. Thus, $\{m_1 : \Delta_{(m_1,m_2)}(x_1', x_2) \geq \Delta\} \subseteq \{m_1 : \Delta_{(m_1,m_2)}(x_1, x_2) \geq \Delta\}$. Hence, $1 - T_1(\Delta|x_1', x_2) \leq 1 - T_1(\Delta|x_1, x_2)$, or $T_1(\Delta|x_1', x_2) \geq T_1(\Delta|x_1, x_2)$. As this holds for all larger Δ, we obtain $\int_{\Delta}^{1} T_1(\Delta'|x_1, x_2)d\Delta' \leq \int_{\Delta}^{1} T_1(\Delta'|x_1', x_2)d\Delta'$. Integrating over all m_2, we have $\int_{\Delta}^{1} T(\Delta'|x_1, x_2)d\Delta' \leq \int_{\Delta}^{1} T(\Delta'|x_1', x_2)d\Delta'$. Thus, $\int_{-1}^{\Delta} T(\Delta'|x_1, x_2)d\Delta' \geq \int_{-1}^{\Delta} T(\Delta'|x_1', x_2)d\Delta'$ as $T(\Delta|x_1, x_2)$ and $T(\Delta|x_1', x_2)$ have the same mean by Step 1.

Hence, $T(\Delta|x_1, x_2)$ is a mean-preserving spread of $T(\Delta|x_1', x_2)$. From here, noting that

$$\frac{\partial V(x, \phi)}{\partial \phi} = \int_{\boldsymbol{m}} (-4h(\Delta_{\boldsymbol{m}}(x)))g(\boldsymbol{m})d\boldsymbol{m} = \int_{\text{supp}(T(x))} -4h(\Delta)dT(\Delta|x)$$

And $-4h(\cdot)$ is concave by assumption, we have that $\frac{\partial V(x,\phi)}{\partial \phi}$ is decreasing in x_1. By symmetry, this implies $\frac{\partial V(x,\phi)}{\partial \phi}$ is decreasing in x_2. Hence, $\frac{\partial V(x,\phi)}{\partial \phi}$ is decreasing in x and schur-concave in x. Remark (1) then implies $X^*(\phi)$ is non-increasing with respect to $\succ_w^M$ in ϕ. $\square$

Proof of Proposition 7 Our proof mirrors that used in Proposition 6. We first identify the equilibrium actions of each receiver. Take any $\boldsymbol{m} \in M^2$. Label $\nabla_{\boldsymbol{m}}(x) = a^*_{\boldsymbol{m},1}(x) + a^*_{\boldsymbol{m},2}(x)$. By each receiver's first-order condition, we note that $a^*_{\boldsymbol{m},1}(x)$ and $a^*_{\boldsymbol{m},2}(x)$ must satisfy

$$a^*_{\boldsymbol{m},1}(x) = \mathbb{E}[\theta_1|m_1, x_1] + \phi s'(\nabla_{\boldsymbol{m}}(x)), \quad a^*_{\boldsymbol{m},2}(x) = \mathbb{E}[\theta_2|m_2, x_2] + \phi s'(\nabla_{\boldsymbol{m}}(x))$$

Adding the two equations yields

$$\nabla_{\boldsymbol{m}}(x) = \mathbb{E}[\theta_1|m_1, x_1] + \mathbb{E}[\theta_2|m_2, x_2] + 2\phi s'(\nabla_{\boldsymbol{m}}(x)) \tag{22}$$

Through implicit differentiation, we obtain

$$\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_i} = \frac{\mathbb{E}[\theta_i|m_i, 1] - \mathbb{E}[\theta]}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))}, \quad \frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial \phi} = \frac{s'(\nabla_{\boldsymbol{m}}(x))}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))}$$
$$\frac{\partial^2 \nabla_{\boldsymbol{m}}(x)}{\partial x_1 \partial x_2} = \left(\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_1}\right)\left(\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_2}\right)\left(\frac{2\phi s'''(\nabla_{\boldsymbol{m}}(x))}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))}\right)$$
$$\frac{\partial^2 \nabla_{\boldsymbol{m}}(x)}{\partial x_i^2} = (\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_i})^2 \frac{2\phi s'''(\nabla_{\boldsymbol{m}}(x))}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))}$$

Now, let $U_{\boldsymbol{m},1}(x)$ and $U_{\boldsymbol{m},2}(x)$ denote receiver 1 and 2's payoffs conditional on message $\boldsymbol{m}$ and allocation x. Applying the envelope theorem, we note that

$$\frac{\partial U_{\boldsymbol{m},1}(x)}{\partial \phi} = 2s(\nabla_{\boldsymbol{m}}(x)) + 2\phi \frac{\partial a^*_{\boldsymbol{m},2}(x)}{\partial \phi} s'(\nabla_{\boldsymbol{m}}(x))$$
$$\frac{\partial U_{\boldsymbol{m},2}(x)}{\partial \phi} = 2s(\nabla_{\boldsymbol{m}}(x)) + 2\phi \frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} s'(\nabla_{\boldsymbol{m}}(x))$$

Noting that $\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial \phi} = \frac{\partial a^*_{\boldsymbol{m},1}(x)}{\partial \phi} + \frac{\partial a^*_{\boldsymbol{m},2}(x)}{\partial \phi}$ and making appropriate substitutions, we may then write the Sender's conditional expected payoff on message $\boldsymbol{m}$, differentiated with respect to ϕ, as

$$\frac{\partial V_{\boldsymbol{m}}(x,\phi)}{\partial \phi} = \frac{\partial U_{\boldsymbol{m},1}(x)}{\partial \phi} + \frac{\partial U_{\boldsymbol{m},2}(x)}{\partial \phi} = 4\left(s(\nabla_{\boldsymbol{m}}(x)) + \frac{(\phi/2)(s'(\nabla_{\boldsymbol{m}}(x)))^2}{(1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x)))}\right)$$

Now, label $h(\nabla) = (s(\nabla_{\boldsymbol{m}}(x)) + \frac{(\phi/2)(s'(\nabla))^2}{(1-2\phi s''(\nabla))})$, and label the determinants of the first and second principal minors of the hessian of $\frac{\partial V_{\boldsymbol{m}}(x,\phi)}{\partial \phi}$ as $det(H_1)$ and $det(H_2)$ (with

x_i for any $i = 1, 2$ in the first entry) We then obtain

$$
\begin{aligned}
det(H_1) &= \frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial \phi \partial x_i^2} = 4\bigg(\frac{\partial^2 \nabla_{\boldsymbol{m}}(x)}{\partial x_i^2} h'(\nabla_{\boldsymbol{m}}(x)) + h''(\nabla_{\boldsymbol{m}}(x)) \bigg[\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_i}\bigg]^2\bigg) \\
&= 4\bigg[\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_i}\bigg]^2 \underbrace{\bigg(\frac{2\phi s'''(\nabla_{\boldsymbol{m}}(x))}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))} h'(\nabla_{\boldsymbol{m}}(x)) + h''(\nabla_{\boldsymbol{m}}(x))\bigg)}_{\text{Simplifies to (14)}} \leq 0 \\
det(H_2) &= \bigg(\frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial \phi \partial x_1^2}\bigg)\bigg(\frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial \phi \partial x_2^2}\bigg) - \bigg(\frac{\partial^3 V_{\boldsymbol{m}}(x)}{\partial \phi \partial x_1 \partial x_2}\bigg)^2 \\
&= 4\bigg[\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_1}\bigg]^2 \bigg(\frac{2\phi s'''(\nabla_{\boldsymbol{m}}(x))}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))} h'(\nabla_{\boldsymbol{m}}(x)) + h''(\nabla_{\boldsymbol{m}}(x))\bigg) \times \\
&\bigg[\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_2}\bigg]^2 \bigg(\frac{2\phi s'''(\nabla_{\boldsymbol{m}}(x))}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))} h'(\nabla_{\boldsymbol{m}}(x)) + h''(\nabla_{\boldsymbol{m}}(x))\bigg) \\
&- 4\bigg(\bigg[\frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_1} \frac{\partial \nabla_{\boldsymbol{m}}(x)}{\partial x_2}\bigg] \bigg(\frac{2\phi s'''(\nabla_{\boldsymbol{m}}(x))}{1 - 2\phi s''(\nabla_{\boldsymbol{m}}(x))} h'(\nabla_{\boldsymbol{m}}(x)) + h''(\nabla_{\boldsymbol{m}}(x))\bigg)\bigg)^2 = 0
\end{aligned}
$$

Hence, the Hessian is negative semi-definite.

From here, an identical argument to the second half of Proposition 6 can be used to then establish that $\frac{\partial V(x,\phi)}{\partial \phi}$ is decreasing in x. We briefly recap the steps below, referring the reader to the proof of Proposition 6:

1. First, show that for all x, $\mathbb{E}[\nabla_{\boldsymbol{m}}(x)] = 2\mathbb{E}[\theta]$. This follows from taking the unconditional expectation of (22) and rearranging to get

$$
\begin{aligned}
\mathbb{E}[\nabla_{\boldsymbol{m}}(x)] + \mathbb{E}[(-2\phi s'(\nabla_{\boldsymbol{m}}(x)))] &= 2\mathbb{E}[\theta] \\
\mathbb{E}[-\nabla_{\boldsymbol{m}}(x)] + \mathbb{E}[(-2\phi s'(-\nabla_{\boldsymbol{m}}(x)))] &= -2\mathbb{E}[\theta]
\end{aligned}
$$

   Where the latter follows from the symmetry of s' about 0, given s's symmetry about 0, and then applying a related argument to that in Proposition 6, noting that $-s'$ is convex as $s''' \leq 0$ by assumption.

2. Let $T(\nabla | x_1, x_2)$ denote the equilibrium distribution of $\nabla_{\boldsymbol{m}}(x)$. Use the above to show that $T(\nabla | x_1, x_2)$ is a mean-preserving spread of $T(\nabla | x_1', x_2)$ when $x_1 \geq x_1'$, by a similar argument to that in Proposition 6

3. From here, noting that $\frac{\partial V(x,\phi)}{\partial \phi} = \int_{\text{supp}(T(x))} 4h(\nabla)dT(\nabla|x_1,x_2)$ and $h(\cdot)$ is concave, one finds that $\frac{\partial V(x,\phi)}{\partial \phi}$ is decreasing in x_1. By symmetry, it is also decreasing in x_2.

Hence, $\frac{\partial V(x,\phi)}{\partial \phi}$ is decreasing in x and schur-concave in x. Remark (1) then implies $X^*(\phi)$ is non-increasing with respect to $\succ_w^M$ in ϕ. □

Before proceeding further with the proof of the Lemma 1 (and subsequent ones related to Section 4.4), we note by the definition of $\overline{U}$ and $\underline{U}$ that

$$U(x_i) = x_i\overline{U}(x_i) + \underline{U}(x_i)$$

With this in mind, we prove the following useful facts about $\overline{U}(\cdot)$ and $\underline{U}(\cdot)$.

Remark 2. *Suppose Assumption 1 holds. Then, $\overline{U}(x_i)$ is strictly increasing in x_i, where $\overline{U}(0) = 0$. Meanwhile, $\underline{U}(x_i)$ is strictly decreasing in x_i, where $\underline{U}(0) = 0$.*

Proof of Remark 2 First, suppose that $x_i = 0$. If so, the receiver takes the default action $a^D \in \arg\max_{a\in A} \int_{\text{supp}(F)} u(\theta,a)dF(\theta)$ under any $m \in M$. This immediately implies $\underline{U}(x_i) = \max_{a\in A} \int_{\text{supp}(F)} u(\theta,a)dF(\theta) = 0$ by Assumption 1. From here,

$$\begin{aligned}\overline{U}(x_i) &= \int_0^1 \int_{\text{supp}(F)} u(\theta,a^D)[g(m|\theta) - g(m)]dF(\theta)dm \\ &= \int_{\text{supp}(F)} u(\theta,a^D) \int_0^1 [g(m|\theta) - g(m)]dmdF(\theta) = 0\end{aligned}$$

Where the interchangability of the integrals follows from Fubini's theorem.

Next, we establish that $\overline{U}(x_i)$ is strictly increasing in x_i. Applying the envelope theorem to $U(x_i)$, which is of the form provided in (2), we notice that $U'(x_i) = \overline{U}(x_i)$. That U is strictly convex then implies that $U''(x_i) = \overline{U}'(x_i) > 0$.

Finally, we establish that $\underline{U}$ is strictly decreasing in x_i. Take $x_i, x_i' \in [0,1]$ with

$x_i > x_i'$. By the definition of $a_m^*(\cdot)$, we note that

$$\begin{aligned}
x_i'\overline{U}(x_i) + \underline{U}(x_i) &= \int_0^1 \left(\int_{\text{supp}(F)} u(\theta, a_m^*(x_i))(x_i'[g(m|\theta) - g(m)] + g(m))dF(\theta) \right) dm \\
&\leq \int_0^1 \max_{a \in [0,1]} \left(\int_{\text{supp}(F)} u(\theta, a)(x_i'[g(m|\theta) - g(m)] + g(m))dF(\theta) \right) dm \\
&= \int_0^1 \left(\int_{\text{supp}(F)} u(\theta, a_m^*(x_i'))(x_i'[g(m|\theta) - g(m)] + g(m))dF(\theta) \right) dm \\
&= x_i'\overline{U}(x_i') + \underline{U}(x_i')
\end{aligned}$$

Hence, $x_i'[\overline{U}(x_i) - \overline{U}(x_i')] + [\underline{U}(x_i) - \underline{U}(x_i')] \leq 0$. From here, that $\overline{U}(x_i) > \overline{U}(x_i')$ implies $\underline{U}(x_i) - \underline{U}(x_i') < 0$. □

Proof of Lemma 1 Take any informative PBE σ^*. Let $\mathcal{N}(\sigma^*)$ be the set of receivers under which $\overline{x}_i(\sigma^*) > 0$, where Remark 2 implies $\overline{U}(\overline{x}_i(\sigma^*)) > 0$ while $\underline{U}(\overline{x}_i(\sigma^*)) < 0$. We note that $\hat{V}(x|\sigma^*, t) = t\sum_{i \in \mathcal{N}(\sigma^*)}(x_i\overline{U}(\overline{x}_i(\sigma^*)) + \underline{U}(\overline{x}_i(\sigma^*)))$. That $\hat{V}(x|\sigma^*, t)$ is linear in x then implies that it is maximized by choosing $x_i = 1$ for some $i \in \max_{i \in \mathcal{N}(\sigma^*)} \overline{U}(\overline{x}_i(\sigma^*))$. Hence, we have for any such i,

$$\max_{x \in X} \hat{V}(x|\sigma^*, t) = t\overline{U}(\overline{x}_i(\sigma^*)) + \sum_{i \in \mathcal{N}(\sigma^*)} \underline{U}(\overline{x}_i(\sigma^*))$$

Which is Sender's gain from opting in. Now, observe that $\max_{x \in X} \hat{V}(x|\sigma^*, t)$ is strictly increasing in t. Furthermore, $\max_{x \in X} \hat{V}(x|\sigma^*, 0) = \sum_{i \in \mathcal{N}(\sigma^*)} \underline{U}(\overline{x}_i(\sigma^*)) < 0$, while we must have $\max_{x \in X} \hat{V}(x|\sigma^*, 1) \geq 0$, for otherwise even the highest type of Sender prefers opting out over in which contradicts σ^* being an informative PBE. Thus, there must exist some threshold type $\overline{t}(\sigma^*) \in (0, 1]$ such that for all $t < \overline{t}(\sigma^*)$, $\max_{x \in X} \hat{V}(x|\sigma^*, t) < 0$ and so the Sender must be opting out under σ^*. In turn, $\mathcal{T}(\sigma^*) = [\overline{t}(\sigma^*), 1]$. □

Proof of Lemma 2 We only prove the first claim, noting that second claim immediate follows from the first. Suppose there exists two receivers i, j under which

$\overline{U}(\overline{x}_i(\sigma^*)) > \overline{U}(\overline{x}_j(\sigma^*))$. By Remark 2, this implies $\overline{x}_i(\sigma^*) > 0$, and so there exists a positive measure of Sender types choosing $x_i > 0$. Furthermore, by Lemma 1, any such type has $t > 0$. By contradiction, suppose that $i \notin \arg\max_{k \in \mathcal{N}} \overline{U}(\overline{x}_k(\sigma^*))$. Then, there exists receiver i' such that $\overline{U}(\overline{x}_{i'}(\sigma^*)) > \overline{U}(\overline{x}_i(\sigma^*))$. If so, then for any such type t, Sender strictly prefers deviating by moving all of the allocation weight from receiver i to receiver i', as this yields a benefit of $x_i t(\overline{U}(\overline{x}_{i'}(\sigma^*)) - \overline{U}(\overline{x}_i(\sigma^*))) > 0$. This contradicts σ^* being a PBE.

Now, by contradiction, suppose that $\overline{x}_j(\sigma^*) > 0$. If so, then $\overline{U}(\overline{x}_j(\sigma^*)) > 0$ by Remark 2. This implies that a positive measure of Sender types are choosing $x_j > 0$ with positive probability. By an identical logic to the above, any such type would prefer deviating by moving all of the allocation weight from receiver j to receiver i. This contradicts σ^* being a PBE. □

Proof of Proposition 8 Take any informative PBE σ^*, and let $\mathcal{N}(\sigma^*)$ denote the set of receivers under which $\overline{x}_i(\sigma^*) > 0$. By Lemma 2, we note that for any $i, j \in \mathcal{N}(\sigma^*)$, $\overline{x}_i(\sigma^*) = \overline{x}_j(\sigma^*) = \overline{x}(\sigma^*)$. By Lemma 2, we know that all Senders with $t \in [\bar{t}(\sigma^*), 1]$ allocate information only among receivers in $\mathcal{N}(\sigma^*)$. Thus, the following equality must hold

$$\underbrace{(1 - Q(\bar{t}(\sigma^*)))|\mathcal{N}(\sigma^*)|\overline{x}(\sigma^*)}_{\text{Expected effective allocation obtained by receivers}} = \underbrace{\int_{\bar{t}(\sigma^*)}^{1} t dQ(t)}_{\text{Total expected effective allocation available to high-type Senders}}$$

Which implies

$$\overline{x}(\sigma^*) = \frac{\int_{\bar{t}(\sigma^*)}^{1} t dQ(t)}{|\mathcal{N}(\sigma^*)|(1 - Q(\bar{t}(\sigma^*)))}$$

From here, that any such receiver obtains the payoff stated in the proposition is simple to verify. Additionally, all receiver $i \notin \mathcal{N}(\sigma^*)$ is never allocated any information by any type of Sender. Thus, all such receivers obtain a payoff of 0.

To prove the second part of Proposition 8, given the symmetry of receivers, we explicitly construct a PBE with any $n \in \{1, ...N\}$ number of receivers allocated information. Fix any such n, and take any group of receivers indexed by $i \in \mathcal{N}$ of size n. Consider a strategy profile σ^* under which

- Senders of types $\tilde{t}$ to 1 allocate $x_i = 1/n$ with probability one equally among receivers in $\mathcal{N}$
- All Senders with types strictly lower than t opt out.

To show that such a strategy profile constitutes a PBE, we first identify a suitable candidate for the threshold type $\tilde{t}$. Observe that if σ^* is to be a PBE, then the expected effective allocation of all $i \in \mathcal{N}$ must be $\frac{\int_t^1 tdQ(t)}{(1-Q(t))n}$. One can then compute the difference for the threshold type t between opting out (and obtaining a payoff of 0) and opting in (and allocating $1/n$ among receivers in $\mathcal{N}$). This is given by

$$t\overline{U}(\frac{\int_t^1 t'dQ(t')}{(1-Q(t))n}) + n\underline{U}(\frac{\int_t^1 t'dQ(t')}{(1-Q(t))n})$$
$$= \underbrace{\left(t - \frac{\int_t^1 t'dQ(t')}{(1-Q(t))}\right)\overline{U}(\frac{\int_t^1 t'dQ(t')}{(1-Q(t))n})}_{A(t)} + \underbrace{nU(\frac{\int_t^1 t'dQ(t')}{(1-Q(t))n})}_{B(t)}$$

We first note that at $t = 0$, $A(0) + B(0) < 0$. This is as $\int_0^1 tdQ(t) > 0$, which implies $n\underline{U}(\frac{1}{n}\int_0^1 tdQ(t)) < 0$. Meanwhile,

$$\lim_{t\to 1}\left(t - \frac{\int_t^1 t'dQ(t')}{(1-Q(t))}\right)\overline{U}(\frac{\int_t^1 t'dQ(t')}{(1-Q(t))n}) = 0, \quad \lim_{t\to 1} nU(\frac{\int_t^1 t'dQ(t')}{(1-Q(t))n}) > 0$$

The first equality holds as $\lim_{t\to 1}\frac{\int_t^1 t'dQ(t')}{(1-Q(t))} = 1$. Meanwhile, $\overline{U}(x_i)$, being continuous on $[0,1]$ by the twice-differentiability of $U(x_i)$, is bounded above on the interval $[0,1]$. Meanwhile, the second inequality holds as $U(x_i)$ is strictly increasing in x_i and $\frac{\int_t^1 t'dQ(t')}{(1-Q(t))n}$ is strictly increasing in t. Hence, $\lim_{t\to 1}[A(1) + B(1)] > 0$

Thus, given the continuity of $A(t) + B(t)$ in t, there must exist some $\overline{t}(\sigma^*) \in$

$(0,1)$ such that

$$\bar{t}(\sigma^*)\overline{U}(\frac{\int_{\bar{t}(\sigma^*)}^1 t'dQ(t')}{(1-Q(\bar{t}(\sigma^*)))n}) + n(\underline{U}(\frac{\int_{\bar{t}(\sigma^*)}^1 t'dQ(t')}{(1-Q(\bar{t}(\sigma^*)))n}) - \underline{U}) = 0$$

We now show that the strategy profile σ^* described previously but with the threshold type as $\bar{t}(\sigma^*)$ (defined above) constitutes a PBE. Take any $t \geq \bar{t}(\sigma^*)$. Since $\overline{U}(\frac{\int_t^1 tdQ(t)}{(1-Q(t))n}) > 0$, by Lemma 1, any such Sender weakly prefers opting in over opting out. Furthermore, since all receivers $i \in \mathcal{N}$ obtain the same expected effective allocation, any such Sender finds it optimal to allocate $x_i = 1/n$ with probability one amongst all such receivers. Meanwhile, any Senders with $t < \bar{t}(\sigma^*)$ must strictly prefer opting out over in. Thus, σ^* with $\mathcal{T}(\sigma^*) = [\bar{t}(\sigma^*), 1]$ constitutes a PBE. $\square$

Proof of Corollary 5 Take any informative PBE σ^* and $\widetilde{\sigma}^*$ under which $|\mathcal{N}(\sigma^*)| > |\mathcal{N}(\widetilde{\sigma}^*)|$. We first establish the existence of $\hat{t}(\sigma^*)$ and $\tilde{t}(\sigma^*)$. Denote the tangent to U at x_i evaluated at x_i' as $U^T(x_i'|x_i)$. We recall from the main-text discussion that $\hat{V}(\sigma^*|t) = |\mathcal{N}(\sigma^*)|U^T(\frac{t}{|\mathcal{N}(\sigma^*)|}|\frac{\int_{\bar{t}(\sigma^*)}^1 tdQ(t)}{(1-Q(\bar{t}(\sigma^*)))|\mathcal{N}(\sigma^*)|})$. Now, observe that for type $t' := \frac{\int_{\bar{t}(\sigma^*)}^1 tdQ(t)}{(1-Q(\bar{t}(\sigma^*)))}$, $\hat{V}(\sigma^*|t') > |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{t'}{|\mathcal{N}(\sigma^*)|})$. This is as type-$t'$ Sender's payoff under any such PBE is simply

$$\hat{V}(\sigma^*|t') = |\mathcal{N}(\sigma^*)|U^T(\frac{t'}{|\mathcal{N}(\sigma^*)|}|\frac{t'}{|\mathcal{N}(\sigma^*)|}) = |\mathcal{N}(\sigma^*)|U(\frac{t'}{|\mathcal{N}(\sigma^*)|}) > |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{t'}{|\mathcal{N}(\sigma^*)|})$$

Here, the second equality holds by the definition of the tangent. The third holds as $U(\frac{t'}{|\mathcal{N}(\sigma^*)|})$ is a strictly positive number, since by Lemma 1, $t' \geq \bar{t}(\sigma^*) > 0$ and so $U(\frac{t'}{|\mathcal{N}(\sigma^*)|}) > 0$.

As a result, the line $|\mathcal{N}(\sigma^*)|U^T(\frac{t}{|\mathcal{N}(\sigma^*)|}|\frac{t'}{|\mathcal{N}(\sigma^*)|})$ sits strictly above $|\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{t}{|\mathcal{N}(\sigma^*)|})$ at $t = t'$. We further note that at the threshold type under σ^*, $\bar{t}(\sigma^*)$, $\hat{V}(\sigma^*|\bar{t}(\sigma^*)) = |\mathcal{N}(\sigma^*)|U^T(\frac{t}{|\mathcal{N}(\sigma^*)|}|\frac{t'}{|\mathcal{N}(\sigma^*)|}) = 0$, while $|\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{\bar{t}(\sigma^*)}{|\mathcal{N}(\sigma^*)|}) > 0$ given $\bar{t}(\sigma^*) > 0$. Hence, $|\mathcal{N}(\sigma^*)|U^T(\frac{t}{|\mathcal{N}(\sigma^*)|}|\frac{t'}{|\mathcal{N}(\sigma^*)|})$ sits strictly below $|\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{t}{|\mathcal{N}(\sigma^*)|})$ at $t = \bar{t}(\sigma^*)$. Combined with the former being linear in t, while the latter is strictly convex and strictly increasing in t, there exists a unique $\hat{t}(\sigma^*) \in (\bar{t}(\sigma^*), t')$ such that $\hat{V}(\sigma^*|\hat{t}(\sigma^*)) = |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{\hat{t}(\sigma^*)}{|\mathcal{N}(\sigma^*)|})$. Meanwhile, at $t = 1$, we have $\hat{V}(\sigma^*|1) \leq |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{1}{|\mathcal{N}(\sigma^*)|})$. Otherwise, some types strictly benefit from the incomplete information environment over the complete information environment, which cannot be true. By a similar logic, there exists a unique $\widetilde{t}(\sigma^*) \in (t', 1]$ such that $\hat{V}(\sigma^*|\widetilde{t}(\sigma^*)) = |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{\widetilde{t}(\sigma^*)}{|\mathcal{N}(\sigma^*)|})$. Further note that any $t \in (\hat{t}(\sigma^*), \tilde{t}(\sigma^*))$, $\hat{V}(\sigma^*|t) > |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{t}{|\mathcal{N}(\sigma^*)|}) > 0$, and so the type opts-in under σ^*.

From here, we note that under $\widetilde{\sigma}^*$, one has, for any $t \in (\hat{t}(\sigma^*), \tilde{t}(\sigma^*))$, $\hat{V}(\widetilde{\sigma}^*|t) = |\mathcal{N}(\widetilde{\sigma}^*)| \max\{0, U^T(\frac{t}{|\mathcal{N}(\widetilde{\sigma}^*)|}|\overline{x}(\widetilde{\sigma}^*))\}$. We further note that $U^T(\frac{t}{|\mathcal{N}(\widetilde{\sigma}^*)|}|\overline{x}(\widetilde{\sigma}^*))$, as the tangent to $U(x_i)$ at $x_i = \overline{x}(\widetilde{\sigma}^*)$, lies weakly below $U(\frac{t}{|\mathcal{N}(\widetilde{\sigma}^*)|}) \geq 0$ for all $\frac{t}{|\mathcal{N}(\widetilde{\sigma}^*)|} \in [0, 1]$, Thus, we have

$$\hat{V}(\sigma^*|t) > |\mathcal{N}(\widetilde{\sigma}^*)|U(\frac{t}{|\mathcal{N}(\widetilde{\sigma}^*)|}) \geq |\mathcal{N}(\widetilde{\sigma}^*)| \max\{0, U^T(\frac{t}{|\mathcal{N}(\widetilde{\sigma}^*)|}|\overline{x}(\widetilde{\sigma}^*))\} = \hat{V}(\widetilde{\sigma}^*|t)$$

This completes our proof. □